Lecture 6: Duration

Key Concepts and Buzzwords

- Interest Rate Sensitivity
- Dollar Duration
- Duration
- Modified duration,
- Macaulay duration,
- Parallel shift
- Basis points

Readings

- Tuckman, chapters 5 and 6.

Duration

- Loose Definition: The *duration* of a bond is a linear approximation of the *percent* change in its price given a 100 basis point (100bp=1%) change in interest rates.
- For example, a bond with a duration of 7 will gain about 7% in value if interest rates fall 100 bp.
- For zeroes, this measure is easy to define and compute with a formula.
- For securities with fixed cash flows, we must make **assumptions** about how rates shift together.
  - → We shall assume all zero rates move by the same amount.
- To compute duration for other instruments requires further assumptions and numerical estimation.

Other Duration Concepts

- Concept 1: Percent change in the bond's price given 100 bp change in rates
- Concept 2: Average maturity of the bond's cash flows, weighted by present value.
- Concept 3: Holding period over which return from investing in the bond is riskless, or immunized from changes in interest rates.
- Mathematical fact: For a security with fixed cash flows, these turn out to be the same.
- For securities with random cash flows, such as callable bonds, concept 2 doesn't really make sense.
- We'll focus on concept 1.

06-5

Dollar Duration

Start with the notion of dollar duration.

Concept:

$$\text{dollar duration} \approx -\frac{\text{change in dollar value}}{\text{change in interest rates}}$$

Application:

$$\text{change in value} \approx -\text{dollar duration} \times \text{change in rates}$$

Example:

Suppose a bond has a dollar duration of 50,000.
How much will its value change if rates fall 11 bp?
Approx. change in value = -50,000 x (-0.0011) = $55

06-6

Dollar Duration and DV01 / DVBP

DVBP = Dollar Value of a Basis Point

How much will a bond value change if rates change 1 bp?

Approx. change in value = -$dur x change in rates

DVBP = $dur x 0.0001

Example:

Bond with $dur = 50,000 has DVBP = 5.

11 bp rate change causes: -(-11)*DVBP=$55 price change.

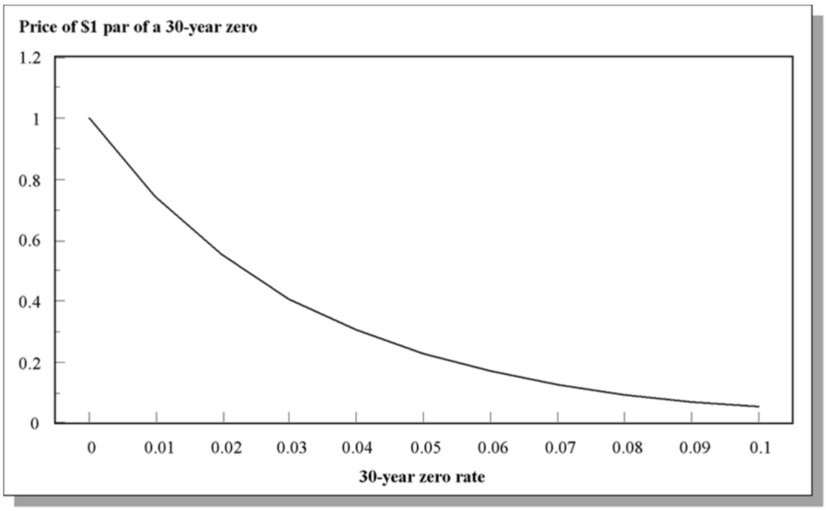
06-7

Computing dollar duration for a zero-coupon bond

- For zero coupon bonds, there is a simple formula relating the zero *price* to the zero *rate*.
- We use this price-rate formula to get a formula for dollar duration.

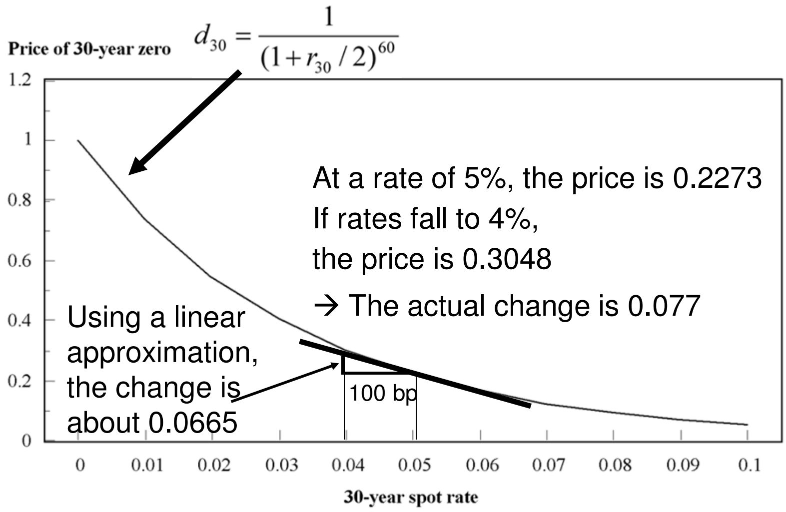
06-8

The Price-Rate Function for a Zero



06-9

The Price-Rate Function for a Zero



06-10

Computing dollar duration for a zero...

•Recall

$$\text{dollar duration} \approx -\frac{\text{change in dollar value}}{\text{change in interest rates}}$$

•By this definition, the dollar duration of the zero is directly related to the slope of the price-rate function.

•Example: The dollar duration of $1 par of a 30-year zero at an interest rate of 5% is 6.65, as illustrated in the last slide: -0.0665/(-0.01)=0.0665/0.01=6.65.

•We can use calculus to compute the slope of the price-rate function and get an explicit formula for the dollar duration of any zero.

06-11

Formula for the Dollar Duration of $1 Par of a Zero-Coupon Bond

$$d_t(r_t) = \frac{1}{(1+r_t/2)^{2t}} = \frac{1}{(1+0.05/2)^{60}} = 0.2273$$

$$d_t'(r_t) = \frac{-t}{(1+r_t/2)^{2t+1}} = \frac{-30}{(1+0.05/2)^{61}} = -6.65$$

To avoid working with negative numbers, change the sign. The dollar duration of $1 par of a t-year zero is

$$\$dur_t = -d_t'(r_t) = \frac{t}{(1+r_t/2)^{2t+1}} = \frac{30}{(1+0.05/2)^{61}} = 6.65$$

06-12

Example

What's the dollar duration of \$1 par of a 1.5-year zero if the 1.5-year discount rate is 5.47%?

$$\frac{t}{(1+r_t/2)^{2t+1}} = \frac{1.5}{(1+0.0547/2)^4} = 1.346535$$

If the rate falls to 5.40%, how much will the price rise? Using the dollar duration approximation, the price will rise by

-1.346535 x (-0.0007) = 0.0009426.

$d_{1.5}(5.47\%) = 0.9222416 \quad d_{1.5}(5.40\%) = 0.9231845$

The actual price rise is 0.0009432

06-13

Dollar Duration of a Portfolio of Fixed Cash Flows

- Consider the dollar sensitivity of a portfolio to a change in interest rates.
- Remember that the portfolio value is a function of all of the different zero rates associated with its cash flows.
- For simplicity, we will approximate the change in the portfolio value assuming *all rates change by the same amount*.
- In other words, we will measure the sensitivity of the portfolio value to a ***parallel shift*** in interest rates.
- How useful will this measure be?
- Of course, rates do not always change by exactly the same amount, but they do tend to move together.

06-14

Dollar duration for a portfolio of fixed cash flows

Suppose a portfolio (or bond) has cash flows *k1, k2,* ... at times *t1, t2,....*
Its value is the sum of the values of the components:

$$V = k_1 \times d_{t_1} + k_2 \times d_{t_2} + \ldots$$

If rates change, its value will change by the sum of the changes in value of the components:

$$\Delta V = k_1 \times \Delta d_{t_1} + k_2 \times \Delta d_{t_2} + \ldots$$

We can approximate the change in each zero price using its dollar duration: $\Delta d_t \approx -\$dur_t \times \Delta r_t$

06-15

Dollar duration for a portfolio of fixed cash flows...

The approximate change in the portfolio value is:

$$\Delta V \approx -(k_1 \times \$dur_{t_1} \times \Delta r_{t_1} + k_2 \times \$dur_{t_2} \times \Delta r_{t_2} + \ldots)$$

Suppose all rate changes are the same.
That is, the yield curve makes a *parallel shift*:

$$\Delta r_{t_1} = \Delta r_{t_2} = \Delta r_{t_3} = \ldots = \Delta r$$

Then the portfolio value change is:

$$\Delta V \approx -(k_1 \times \$dur_{t_1} + k_2 \times \$dur_{t_2} + \ldots) \times \Delta r$$

06-16

Dollar duration for a portfolio of fixed cash flows...

Then the portfolio dollar duration is:

$$\text{portfolio \$dur} \approx -\frac{\text{change in value}}{\text{change in rates}} = -\frac{\Delta V}{\Delta r}$$

$$\Delta V \approx -(k_1 \times \$dur_{t_1} + k_2 \times \$dur_{t_2} + \ldots) \times \Delta r$$

$$\Rightarrow \text{ portfolio \$dur} \approx -\frac{\Delta V}{\Delta r} \approx k_1 \times \$dur_{t_1} + k_2 \times \$dur_{t_2} + \ldots$$

In other words, the dollar duration of the portfolio is the sum of the dollar durations of its cash flows:

$$\text{portfolio \$dur} = k_1 \times \$dur_{t_1} + k_2 \times \$dur_{t_2} + \ldots$$

06-17

Example

- What is the dollar duration of a portfolio consisting of $500 par of the 1.5-year zero and $100 par of the 30-year zero?
- (500 x 1.35) + (100 x 6.65) = 1340
- This means the portfolio value will change about $13.40 for every 100 basis point shift in interest rates.
- Why?
  - Each 100 bp change in the 1.5-year rate changes the value of the 1.5-year zero about 500 x 1.35 x 0.01 = 6.75.
  - Each 100 bp change in the 30-year rate changes the value of the 30-year zero about 100 x 6.65 x 0.01 = 6.65.
  - The total portfolio change is about

    6.75 + 6.65 = 13.40 = ((500 x 1.35) + (100 x 6.65)) x 0.01 = 1340 x 0.01.

06-18

Duration

Duration approximates the ***percent*** change in price for a 100 basis point change in rates:

$$\begin{aligned}&\text{duration} \approx \\ &\text{percent change in value per 100 bp change in rates} \\ &= \frac{\text{dollar change in value per 100 bp}}{\text{initial value}} \times 100 \\ &= \frac{\text{dollar duration} \times 0.01}{\text{initial value}} \times 100 \\ \rightarrow\ &= \frac{\text{dollar duration}}{\text{initial value}}\end{aligned}$$

06-19

Duration for a zero

$$\text{duration} = \frac{\text{dollar duration}}{\text{price}}$$

The duration of a t-year zero is:

$$\text{duration} = \frac{\dfrac{t}{(1+r_t/2)^{2t+1}}}{\dfrac{1}{(1+r_t/2)^{2t}}} = \frac{t}{(1+r_t/2)}$$

Notice that the duration of a zero is just slightly less than its maturity.

06-20

Example: 1.5-year zero

- At an interest rate of 5.47%, the duration of the 1.5-year zero is

$$\text{duration} = \frac{\text{dollar duration}}{\text{price}} = \frac{1.3465}{0.9222} = 1.46$$

$$\Leftrightarrow \text{duration} = \frac{t}{(1 + r_t / 2)} = \frac{1.5}{1 + 0.0547/2} = 1.46$$

If rates rise 100 basis points to 6.47%, the price falls about 1.46% from 0.9222 to 0.9087

06-21

Example: 30-year zero

At an interest rate of 5%, the duration of the 30-year zero is

$$\text{duration} = \frac{\text{dollar duration}}{\text{price}} = \frac{6.65}{0.2273} = 29.26$$

$$\Leftrightarrow \text{duration} = \frac{t}{(1 + r_t / 2)} = \frac{30}{1 + 0.05/2} = 29.26$$

If rates fall 100 basis points to 4%, the price rises about 29.26% from 0.2273 to about 0.2938.

If rates fall only 50 bp to 4.5%, the price rises only half as much, about 14.63% to about 0.2606.

06-22

Duration of a Portfolio

- Just as with a zero, the duration of a portfolio is its dollar duration divided by its market value.
- The duration gives the *percent* change in value for each 100 basis point change in all rates.

$$\text{duration} = \frac{\text{dollar duration}}{\text{value}} = \frac{k_1 \times \$dur_{t_1} + k_2 \times \$dur_{t_2} + \ldots}{k_1 \times d_{t_1} + k_2 \times d_{t2} + \ldots}$$

06-23

Example

The duration of the portfolio consisting of $500 par of the 1.5-year zero and $100 par of the 30-year zero is

$$\text{duration} = \frac{\text{dollar duration}}{\text{market value}} = \frac{1340}{483.85} = 2.8$$

This means that the portfolio value will change about 2.8% for every 100 basis point change in interest rates.

06-24

Duration of a Portfolio as Average Duration

We can think of the portfolio duration as the average of the durations of the individual cash flows, weighted by their present value or "market" value.

Recall that the dollar duration of each zero is its duration times its price: $\$dur_t = d_t \times dur_t$

So the portfolio duration is

$$\text{portfolio dur} = \frac{k_1 \times d_{t_1} \times dur_{t_1} + k_2 \times d_{t_2} \times dur_{t_2} + \ldots}{k_1 \times d_{t_1} + k_2 \times d_{t_2} + \ldots}$$

$$\Rightarrow \text{portfolio dur} = w_1 \times dur_{t_1} + w_2 \times dur_{t_2} + \ldots.$$

$$\text{where } w_i = \frac{k_i \times d_{t_i}}{k_1 \times d_{t_1} + k_2 \times d_{t_2} + \ldots} \text{ is the pv weight of cash flow i}$$

06-25

Duration of a Portfolio

$$\text{portfolio dur} = \frac{k_1 \times d_{t_1} \times dur_{t_1} + k_2 \times d_{t_2} \times dur_{t_2} + \ldots}{k_1 \times d_{t_1} + k_2 \times d_{t_2} + \ldots}$$

$\Rightarrow$

$$\text{duration} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1 + r_{t_j}/2)^{2t_j}} \times \frac{t_j}{(1 + r_{t_j}/2)}}{\sum_{j=1}^{n} \frac{K_j}{(1 + r_{t_j}/2)^{2t_j}}}$$

06-26

Example

Recall the portfolio consisting of $500 par of the 1.5-year zero and $100 par of the 30-year zero.

The market value of the 1.5-year zero is 500 x 0.92224 = $461.12. Its duration is 1.46.

The market value of the 30-year zero is 100 x 0.2273 = $22.73. Its duration is 29.26.

The duration of the portfolio is:

$$\frac{(\$461.12 \times 1.46) + (\$22.73 \times 29.26)}{\$461.12 + \$22.73} = 2.8$$

06-27

Example...

In terms of the market value weights, the duration of the portfolio is as follows:

$$w_1 = \frac{461.12}{483.85} = 95.3\%,\ w_2 = \frac{22.73}{483.85} = 4.7\%$$

$$\text{portfolio duration} = 0.953 \times 1.46 + 0.047 \times 29.26 = 2.8$$

06-28

Modified Duration

- In practice, people compute what's called the ***modified duration*** of a security by using the security's **yield** instead of the different zero rates associated with each cash flow.

$$\text{modified duration} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}} \times \frac{t_j}{(1+y/2)}}{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}}}$$

- The modified duration of a portfolio is the average modified duration of its securities weighted by their market value.

06-29

Example

Original data

Par	Coupon (%)	Maturity (years)	Yield (%)	Market Value	Duration	Dollar Duration
1	0	0.5	5.54	0.973047	0.486523	0.473410
1	0	1.0	5.45	0.947649	0.973473	0.922511
100	6.5	1.0	5.451431	101.0072	0.958227	96.787842

Modified duration of the coupon bond

Par	Coupon (%)	Maturity (years)	Yield (%)	Market Value	Duration	Dollar Duration
1	0	0.5	5.451431	0.973466	0.486733	0.473818
1	0	1.0	5.451431	0.947636	0.973466	0.922492
100	6.5	1.0	5.451431	101.0072	0.958221	96.787179

06-30

Modified Duration...

- Computing the bond's modified duration is like computing its duration **assuming the yield curve is flat**, i.e., that all rates are equal to the bond's yield.
- When the yield curve is not flat, using security yield instead of individual zero rates creates a difference that is usually slight.

06-31

Modified Duration: Inconsistency vs. Practicality

- Using modified duration can lead to logical inconsistencies if the yield curve is not flat.
  - If the yield curve is not flat, then two portfolios with identical cash flows can have slightly different modified durations. For example, think of a coupon bond and the corresponding portfolio of zeroes.
    - In the previous example, the coupon bond had a modified duration of 0.958221.
    - The replicating portfolio of zeroes would have a modified duration of 0.958227.
- BUT: If zero rates are not readily available, modified duration is easier to compute.

06-32

Finally: Macaulay Duration

The first measure of duration was developed by Frederick Macaulay in 1938:

$$\text{Macaulay duration} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}} \times t_j}{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}}} = \text{modified duration} \times (1+\text{y}/2)$$

Note that the Macaulay duration of a *t*-year zero is just its time to maturity, *t*.

The Macaulay duration of a security is the average maturity of each cash flow weighted by the cash flow's present value at the yield on the security.

This gives an **intuitive way to guess the interest rate sensitivity of a bond.**

06-33