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An Equilibrium Model of Nominal Bond Prices with Inflation-Output Correlation and Stochastic Volatility

THE EMPIRICAL EVIDENCE addressed in this study of the term structure of interest rates focuses on the link between interest rates and the role of the time series properties of inflation and consumption growth. In particular, a negative correlation between shocks to inflation and consumption growth, as well as negative correlation between expected changes, is assumed. We also observe time variations in the conditional variance of inflation. To what extent can we explain the dynamic properties of real and nominal interest rates as a reflection of these assumed exogenous processes,within the representative agent paradigm? Can we explain evidence of autoregressive conditional heteroskedasticity of excess holding-period returns (identified as time-varying risk) and time-varying expected excess returns (identified as commensurate time-varying risk premium) as a reflection of the assumed exogenous processes of inflation and consumption? We show that the autoregressive stochastic volatility of inflation affects the time series properties of the risk premia implicit in bond prices.[1] We also show that the codependence of con-

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1. Shiller, Campbell, and Schoenholtz (1983), Mankiw and Summers (1984), and others discuss rejections of the expectations hypothesis of interest rates [see Shiller (1990) for a survey]. The variations in risk and return implicit in bond returns are analyzed by Engle, Lillien, and Robbins (1987). They specify an ARCH-M (autoregressive conditional heteroskedasticity with mean effects) process for bond returns in which the conditional variance of excess bond returns varies through time and is positively related to

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sumption growth and inflation which we can allow for in our model gives rise to the widely documented codependence (both ex ante and ex post) between real rates and inflation.[2]

Nominal bond prices are assumed to be the conditional expectation of the nominal marginal rates of substitution. Assuming the existence of a representative agent with a time-separable power utility function, the nominal marginal rate of substitution between period t and period $t + \tau$ is $\beta^{\tau} G_{t,t+\tau}{}^{-\gamma} I_{t,t+\tau}{}^{-1}$. Here $G_{t,t+\tau} = c_{t+\tau}/c_t$ is the rate of growth in consumption, $I_{t,t+\tau} = p_{t+\tau}/p_t$ is the rate of inflation, β is the time discount parameter and γ is the relative risk aversion parameter. The time t nominal price of a bond paying \$1 at $t + \tau$ is

$$Q_t^{\tau} = E_t[\beta^{\tau} G_{t,t+\tau}{}^{-\gamma} I_{t,t+\tau}{}^{-1}] \, .$$

In order for us to be able to price nominal bonds in this framework we need to make explicit assumptions about the joint processes of inflation and consumption. We assume that the log of inflation and the log of consumption growth follow a vector autoregressive (VAR) process, which allows for codependencies of the processes and their innovations.The traditional VAR is extended to allow for stochastic shifts in the conditional volatility of inflation, driven by an unobservable autoregressive process. This VAR-SV (vector autoregressive process with stochastic volatility) specification is motivated partly by empirical evidence and partly by a theoretical discussion. The parameters of the VAR-SV model are estimated using quarterly data on inflation and consumption of nondurables and services.

Given the estimated parameters of the VAR-SV model and a set of preference-related parameters (β and γ), we can price nominal and real bonds, using the first-order conditions of the representative agent. Under our distributional assumptions, however, we cannot offer closed-form solutions, and bond prices are approximated using a discrete state-space methodology for solving stochastic integral equations. The method is an extension of the Gaussian Quadrature Method used by Tauchen and Hussey (1991),[3] and extended here for the stochastic volatility case. Using this solution method we can analyze static and dynamic properties of the term structure, and the dependence of these properties on each state variable. We can use closed-form solutions only for spot rates, where conditional lognormality holds. The preference-related parameters are calibrated by matching moments of observed bond prices to moments of theoretical bond prices. Theoretical bond prices are gen-

the conditional expected excess return, as expected in a risk-averse economy. Pagan and Hong (1988) relax the common approach of parameterizing the relation between risk premia and conditioning information. Their nonparametric approach exhibits significant improvement over parametric approaches.

2. This effect is commonly called the Tobin-Mundell effect: expected real returns and expected inflation are negatively correlated. The Tobin-Mundell model attributes the negative link to the fact that high expected inflation raises the opportunity cost of money. Higher demand for bonds reduces the expected real rate, hence increasing capital expenditure. See also Fama and Gibbons (1982) for alternative interpretations.

3. The Gaussian Quadrature Method specifies the best discretization scheme ("best" in a statistical sense) of the state variables, given a finite number of discretization states. For each discrete state we obtain a value for each of the three state variables and the transition probability matrix.

erated using the parameters estimated for the VAR-SV model and preference parameters.

Notice that the method of analysis here enables us to distinguish between expected and unexpected changes in the state variables. Given the equilibrium model these translate into expected and unexpected changes in the real and nominal term structure. We can therefore also distinguish between variations in the risk premium and shocks.

We provide evidence that the empirically documented codependence of inflation and consumption growth is important in the determination of nominal and real interest rates. For example, when the spot rate is written as a function of the state variables—consumption growth, inflation and inflation volatility—the cross term of inflation explaining consumption growth accounts for 22 percent of nominal bond prices on average. Real rates, the expectation of the real marginal rate of substitution, also depend on the future path of inflation, and therefore on the current state of inflation, in a similar way. The state of inflation has an additional opposite and direct effect only on nominal bond prices that accounts on average for 36 percent of bond prices. In addition, a model that allows for inflation-output correlation and stochastic volatility is shown to perform better than a model without these features. To show this we reestimate the VAR-SV model with time invariant volatility (the log-linear model) and with inflation-output independence (the restricted VAR-SV model). We observe that not only are the restrictions counterfactual given inflation and consumption data, but also that the implied moments of yields (means, variances, and autocorrelations) across maturities are matched better by the unrestricted model.

The factor driving inflation volatility becomes important when analyzing excess bond returns. The reason is simple: the variations and the average magnitude of the premia are small relative to bond prices, but not small relative to excess holding-period returns. For example, we show that in a certain state of the world the price of a three-month bond is \$.9843 per \$1 of face value, and the three- to six-month premium is \$.0014. We find that the state variable governing inflation volatility becomes important when analyzing risk premia, and we can account for the range of magnitudes taken by the risk premia. We can account for the variability of risk premia, but not for its average magnitude. In the data, the average risk premia, while small, is positive and significantly different from zero. In our model, its average magnitude is indistinguishable from zero, consistent with the "equity premium puzzle" for the case of bonds.

Although the state of inflation volatility is not directly observable, we obtain a measure of the volatility implied by inflation, consumption, and the spot rate using estimated parameters. Implied volatility is often undefined (since it is the positive solution of a quadratic), unless the relative risk aversion parameter is increased above its estimate (3.584). With a relative risk aversion parameter as high as ten, the only two periods when the implied volatility is not defined are 1973Q4–1974Q4 and 1979Q4–1982Q4. We provide an interpretation for these observations, looking at the wedge between spot rates and the effects of the conditional means of consumption growth and inflation on nominal rates.

How does this model compare to other equilibrium models of the nominal term structure of interest rates? The equilibrium model developed by Cox, Ingersoll, and Ross (1985) (CIR henceforth) exhibits time-varying expected excess-holding-period returns on bonds, driven by assumptions about the time series properties of fundamentals. Since the state, the instantaneous real riskless rate on an instantaneous bond, follows a square root diffusion process, it exhibits level-dependent conditional heteroskedasticity. CIR extend the model to the pricing of nominal bonds by augmenting the state space with an *orthogonal* inflation process.[4] The inflation independence assumption enables CIR to get a closed-form solution for nominal assets. They point out that giving a role to money would necessitate additional features in the model. However, some aspects of inflation can be captured by the addition of a price-level variable, potentially statistically correlated (with no causality statements made explicit) with changes in real consumption. Here, the existence of conditional and unconditional correlation patterns between inflation and output are accounted for in reduced forms. We do not endogenize this phenomenon, but still try to account for its effect on nominal (real) bond prices through the effect on the conditional expectations on the nominal (real) marginal rates of substitution.

Pennacchi (1991) estimates a *homoskedastic* VAR of instantaneous real rates and expected inflation, using bond prices and forecasts of inflation, within the general CIR framework. His findings regarding the importance of codependencies between real and nominal processes in the determination of interest rates are consistent with ours. The scope of his analysis is limited by the homoskedasticity assumption, which implies time *invariant* risk premia. Results in this paper imply that this is not a severe limitation since interest rate dynamics can be captured by the inflation and consumption processes, and inflation volatility turns out to be important only for the (much smaller in magnitude) excess return series.

The specification of inflation volatility as stochastic (rather than, say, that of consumption growth) is not arbitrary. Separating the nominal rate into ex post real rates and realized inflation generates evidence consistent with inflation being the source of conditional heteroskedasticity in nominal excess returns. In particular, Bollerslev (1988) shows that inflation and nominal rate series are cointegrated in variance, but not in means, therefore the forcing process(es) driving the conditional variances of (first differences of) inflation and nominal rates appear to be common. In contrast,

4. Stambaugh (1988) tests one implied restriction of the CIR model, that excess forward rates should be linear predictors of excess-holding-period returns (the test is of an implied rank condition on the forward rate coefficients). The evidence varies across data sets, with evidence of as many factors as maturities in one data set, and two to four factors in another data set. Identifying more than two factors in the term structure is not evidence against the CIR model, if specified as a multifactor model, as by Chen and Scott (1990). There is, nevertheless, strong empirical evidence against the distributional assumptions of the CIR model. Specifically, Heston (1988) finds strong evidence against the form of the nonlinear cross-equation restrictions implied by the model for excess-holding-period returns. Pearson and Sun (1989) reject the two-factor version of the CIR model by showing that forward rates convey information regarding the prediction error implied by the difference between the model's implied expected holding-period return and the realized holding-period return.

ex post real rates show no significant conditional heteroskedasticity.[5] Sun (1987) tests the two inflation models in CIR, and concludes that the second model, in which both the speed of adjustment and the relative variance increase with the level of expected inflation, performs better than the first, which assumes a constant speed of reversion toward the long-run mean. We do not impose the link of conditional volatility to levels, but let them be statistically related, by specifying as a free parameter the correlation coefficient of shocks to inflation and inflation volatility.

The paper proceeds as follows: In section 1 we introduce the pricing framework and the VAR-SV model. In section 2 we explore empirically the appropriateness of the distributional assumptions we make, and estimate the VAR-SV model. In section 3 we calibrate preference-related parameters (given the law of motion of inflation and consumption) using bond data. We investigate the relevance of various properties of the VAR-SV model for nominal and real interest rates, and focus the discussion on factors determining bond prices and risk premia. Section 4 concludes the paper.

1. THE MODEL

1a. Valuation of Nominal Bonds

Consider an economy with a set of investors who maximize expected utility subject to sequential budget constraints (see Lucas 1978 and Mehra and Prescott 1985). Under certain regularity conditions, the agents can be replaced by a representative agent with an additively separable utility function

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \ , \tag{1}$$

where c_t is consumption at time t and β is the time discount factor. Assets can be priced by the well-known first-order conditions of optimization

$$p_t u'(c_t) = \beta \, E_t \, [u'(c_{t+1})(p_{t+1} + y_{t+1})] \ , \tag{2}$$

where p_t and y_t are the asset's real price and dividend, respectively. The real price of a τ period real riskless pure discount bond q_t^τ is solved for by repeated substi tution into (2), which yields

$$q_t^\tau = \beta^\tau E_t \left[\frac{u'(c_{t+\tau})}{u'(c_t)} \cdot 1 \right] . \tag{3}$$

Let p_t be the monetary cost of a unit of consumption at date t. The time t nominal price of a nominal bond which pays one dollar at $t + \tau$ is

5. The analysis is implemented using quarterly data on T-bill returns and the implicit price deflater. AR(6) is tested against AR(6)-GARCH(1,1).

$$Q_t^\tau = \beta^\tau E_t \left[\frac{u'(c_{t+\tau})}{u'(c_t)} \cdot \frac{p_t}{p_{t+\tau}} \right] . \tag{4}$$

To get (4), consider the real price of a security paying zero dividends at $t + 1, \ldots,$ $t + \tau - 1$, and $1/p_{t+\tau}$ units of the consumption good at $t + \tau$. To get the time t nominal price of such a claim, we then multiply both sides of (3) by p_t.

The line of argument used here to motivate (4) assumes that the economy is a pure exchange economy. Markets are *effectively complete* (Ingersoll 1987, chap. 8), since all investors are assumed identical in tastes, endowments, and beliefs. Markets are therefore Pareto efficient. Inflation enters as a numeraire, specified exogenously. Nevertheless, nothing precludes the possibility that inflation is correlated with "real variables," for example, consumption (output). A number of authors, for example, Constantinides (1990) and Pennacchi (1991), follow a similar path. Constantinides (1990, p. 1) discusses an asset-pricing model similar to (4) [(5) there] and notes,

> The model may be motivated by the existence of a pricing operator guaranteed to exist by the absence of arbitrage in a frictionless market. Alternatively the model may be obtained as the equilibrium in a representative-consumer exchange or production economy. The instruments which are the exogenous forcing processes may be related to the per capita consumption and price level processes.

Assume that the agent's utility function exhibits constant relative risk aversion (CRRA). The real marginal rate of substitution between time t consumption and time $t + \tau$ consumption is, therefore, $\beta^\tau(c_{t+\tau}/c_t)^{-\gamma}$ where $\gamma \geq 0$ is the CRRA parameter. Let $G_{t,t+\tau}$ be the rate of growth in consumption $c_{t+\tau}/c_t$, and let $I_{t,t+\tau}$ be the price-level growth (inflation) $p_{t+\tau}/p_t$. The nominal price of a nominal zero coupon bond is then

$$Q_t^\tau = E_t\,[\beta^\tau G_{t,t+\tau}{}^{-\gamma}] \cdot E_t\,[I_{t,t+\tau}{}^{-1}] + \mathrm{cov}_t\,(\beta^\tau G_{t,t+\tau}{}^{-\gamma}; I_{t,t+\tau}{}^{-1}) . \tag{5}$$

This is the basic valuation equation which underlies the determination of nominal interest rates.

1b. Distributional Assumptions

A complete model of the dynamics of the term structure of interest rates necessitates a specific set of distributional assumptions on the driving processes. Denote

$$\mathbf{x}_{t+1} \equiv \begin{bmatrix} \ln(G_{t+1}) \\ \ln(I_{t+1}) \end{bmatrix} \equiv \begin{bmatrix} g_{t+1} \\ i_{t+1} \end{bmatrix} \tag{6}$$

where the subscript $t + 1$ replaces the time subscript $t, t + 1$ henceforth. Suppose x_{t+1} follows a VAR(L) process,

$$\mathbf{x}_{t+1} = \mathbf{B} + \sum_{l=0}^{L-1} \mathbf{A}_l \mathbf{x}_{t-l} + \varepsilon_{t+1} . \tag{7}$$

Here $\varepsilon_{t+1} \equiv [\epsilon^g_{t+1}, \epsilon^i_{t+1}]'$ are the innovations in consumption growth and inflation. We assume $E_t[\varepsilon_{t+1}] = 0$ and $\text{var}_t[\varepsilon_{t+1}] = \boldsymbol{\Sigma}_t$.

The Log-Linear Model

The Log-Linear Model (LLM) is given by restricting the covariance matrix to be time invariant: $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}, \forall t$. Various versions of the model and its implied restrictions are studied by Hansen and Singleton (1983), Breeden (1985), and others. One such restriction implied by log-linearity is that the expected excess-holding-period return should be fixed (for example, Boudoukh 1991). Another is that the forward rate should equal the expected spot rate, plus a time-invariant (but potentially maturity-dependent) premium. Rejections of these restrictions are prevalent. One common interpretation of such rejections is that the premiums are time varying, and in particular, are autocorrelated. The link to time-varying second moments of fundamentals is therefore plausible.

Inflation Volatility as a Stochastic Factor—The VAR-SV Model

Empirical evidence suggests that real variables (such as output/consumption) and inflation are not independent. This holds true for expected as well as unexpected changes in these variables. Such evidence is offered here, as well as in section 2, where we suggest a VAR specification for the joint time series properties of consumption and inflation. Moreover, we examine empirical evidence that supports modeling inflation volatility as time varying. Inflation volatility is modeled as stochastic and autoregressive, driven by an unobservable factor, denoted v_t, whose innovations may be correlated with innovations to inflation.

The VAR-SV Model:

$$g_{t+1} = b_1 + a_{11}g_t + a_{12}i_t + \epsilon^g_{t+1} , \tag{8}$$

$$i_{t+1} = b_2 + a_{21}g_t + a_{22}i_t + \epsilon^i_{t+1} , \tag{9}$$

$$v_{t+1} = b_3 + a_{33}v_t + \epsilon^v_{t+1} , \tag{10}$$

where

$$\begin{bmatrix} \epsilon^g_{t+1} \\ \epsilon^i_{t+1} \\ \epsilon^v_{t+1} \end{bmatrix} | \Phi_t \sim \mathrm{N}\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} \sigma^2_g & \rho_{gi}\sigma_g \exp v_t & 0 \\ \rho_{gi}\sigma_g \exp v_t & \exp 2v_t & \rho_{iv}\sigma_v \exp v_t \\ 0 & \rho_{iv}\sigma_v \exp v_t & \sigma^2_v \end{pmatrix} \right] , \tag{11}$$

where ρ_{gi} and ρ_{iv} are the correlation coefficients.[6]

The specification in (8)–(11) is an extension of a VAR setup of the stochastic

6. Note that we could write (9) instead as $i_{t+1} = b_2 + a_{21}g_t + a_{22}i_t + \exp(v_t)\epsilon^i_{t+1}{}'$ (that is $\epsilon^i_{t+1} = \exp(v_t)\epsilon^i_{t+1}{}'$) with the appropriate changes to the covariance matrix in (11).

volatility specification in Melino and Turnbull (1990). The state space here includes the state variable of inflation volatility: $V_t \equiv \exp(v_t)$. The conditional covariance $\mathrm{cov}_t[i_{t+1}, g_{t+1}]$ is the product of a time-invariant correlation coefficient, ρ_{gi}, a time-invariant standard deviation of shocks to consumption growth, σ_g, and a time-varying autoregressive standard deviation of shocks to inflation, V_t. Since the state variable of inflation volatility is autoregressive [see (10)], the covariance term $\mathrm{cov}_t[i_{t+1}, g_{t+1}] = \rho_{gi}\sigma_g V_t$ is also autoregressive. The volatility state variable is not assumed to be directly observable by the econometrician.

Regarding the timing of information about inflation volatility, implied by (10) and (11), notice that uncertainty about next period is summarized by a bivariate log-normal distribution of consumption growth and inflation, while next period's inflation volatility is known. We could choose to have volatility uncertainty as well, but in either case agents face volatility uncertainty from period $t + 2$ on.

The VAR-SV model departs from the commonly used independence assumption (for example, CIR 1985). The processes here are allowed to be dependent (via nonzero off-diagonal terms of the coefficient matrix). Their shocks are allowed to be dependent as well (via nonzero correlation between shocks to consumption growth and inflation and between inflation and inflation volatility). Empirical evidence presented later is indicative of the statistical significance of these generalizations, and the effect they have for real and nominal interest rates.

In the VAR-SV model the correlation between shocks to inflation and inflation volatility is potentially nonzero, in contrast with an ARCH/GARCH specification, and in line with a signed EGARCH. In fact, the special case $\rho_{iv} = 0$ is analogous to the EGARCH specification.[7] The VAR-SV model allows for periods with low but highly uncertain inflation, or high but quite persistent inflation. That such periods are less likely than cases of high and volatile or low and smooth inflation can be captured by ρ_{iv}.

The VAR-SV model suffers from a number of limitations. One is the lag length of the VAR. Another is that inflation's level is related to its volatility via the parameter ρ_{iv} only, and not via a term $\ldots + a_{23}v_t$, hence only unexpected shocks to inflation and inflation volatility are related. Empirical evidence presented later indicates that these simplifications are not a major limitation.

2. PARAMETER ESTIMATION FOR BOND PRICING

In this section we first establish the link between inflation and consumption growth. We show that the off-diagonal terms in the VAR and the correlation of the shocks are statistically significant. We then turn to the estimation of the parameters of the VAR-SV model that will serve us as an input for bond pricing.

7. Nelson and Foster (1991) show that it is possible to discretize the stochastic volatility model in Melino and Turnbull (1990), and write it as a *misspecified* AR(1)-EGARCH. They show that in spite of the misspecification, "correct" estimates of the conditional variance can be obtained, and correct filtering and forecasting can therefore be performed. The state of conditional volatility is observable given its initial value. Such a misspecification does not occur in our discretization, but the state of the volatility remains unobservable.

2a. *Data*

The data are taken from the Citibase database Chapter X: National Income and Product Accounts. We use personal consumption data, including nondurables (GCND, GCND82) and services (GCS, GCS82) in current and 1982 dollars. The series are seasonally adjusted. The implicit price deflator is calculated based on these two series. To calculate the per capita consumption of nondurables and services, we divide the consumption series by the population series (POP). Following Mehra and Prescott (1985) and others, the series are assumed to be stationary in growth. Although the data available to us start in 1948, empirical studies commonly skip the first few years of the postwar era. The year 1953 is a common choice as a starting date due to changes in the sampling procedure of consumption, the Fed-Treasury accord, and price controls. All of these changes may have perverse effects on the variables of interest prior to 1953.

2b. *VAR and Conditional Heteroskedasticity*

Table 1 reports the results of vector autoregressions of lag-lengths one and two. The standard errors are all heteroskedasticity consistent. In the VAR(1) the R^2 of consumption growth is 8.95 percent, while that of inflation is 73.2 percent. Inflation is highly autoregressive and largely predictable, while variations in consumption growth can be explained only to a much lesser extent using lagged inflation and consumption growth. Regarding the cross-effects, notice that inflation does not en-

TABLE 1

LOG CONSUMPTION GROWTH AND LOG INFLATION VECTOR AUTOREGRESSION

VAR(1):

	Intercept	g_t	g_{t-1}	i_t	i_{t-1}	R^2	HET1	HET2	HET3
g_{t+1}	.005	.189	—	−.121	—	.089	.637	.923	.235
	(.001)	(.091)		(.083)			(.425)	(.819)	(.889)
i_{t+1}	.76e-4	.229	—	.899	—	.732	3.332	5.388	2.871
	(.76e-3)	(.067)		(.051)			(.0679)	(.145)	(.238)

$\chi^2(a_{12}^{(1)}, a_{21}^{(1)}) = 13.082(.001)$

$\chi^2(a_{11}^{(1)}, a_{12}^{(1)}) = 11.379(.003)$

$\chi^2(a_{21}^{(1)}, a_{22}^{(1)}) = 309.838(.000)$

VAR(2):

	Intercept	g_t	g_{t-1}	i_t	i_{t-1}	R^2	HET1	HET2	HET3
g_{t+1}	.005	.187	.0169	−.112	−.009e-2	.090	.586	2.780	2.054
	(.001)	(.096)	(.087)	(.131)	(.130)		(.444)	(.734)	(.726)
i_{t+1}	−.009e-4	.144	.169	.591	.361	.761	1.453	8.672	7.928
	(.71e-3)	(.057)	.057	(.090)	(.080)		(.228)	(.123)	(.094)

$\chi^2(a_{12}^{(1)}, a_{12}^{(2)}) = 2.928(.231)$

$\chi^2(a_{21}^{(1)}, a_{21}^{(2)}) = 18.619(.000)$

$\chi^2(a_{12}^{(1)}, a_{12}^{(2)}, a_{21}^{(1)}, a_{21}^{(2)}) = 20.136(.47e - 3)$

Data: 1953Q1–1989Q1 per capita consumption of nondurable goods and services and the implicit price deflator. Source: NIPA. Heteroskedasticity-consistent standard errors are in parentheses. HET1 is Engle's (1982) LM test for ARCH. HET2 is a Breusch-Pagan test with lagged consumption growth, lagged inflation, and lagged squared residuals. HET3 is a Breusch-Pagan test without the lagged squared residuals. The exclusion tests are calculated using the covariance matrix of the parameters obtained via GMM.

ter significantly into the regression of consumption growth. In particular, a_{12} is only 1.45 standard deviations away from zero. As for inflation, on the other hand, both lagged consumption growth and lagged inflation enter significantly into the prediction equation. In particular, a_{21} is 3.4 standard deviations away from zero. Point estimates of the cross-effects coincide with common wisdom regarding the relation between the variables. An asymptotic $\chi^2(2)$ test of exclusion of the cross-dependence coefficients, a_{12} and a_{21}, yields a test statistic of 13.083 with a P-value of .00144, indicative of the statistical importance of accounting for these cross-dependencies. No significant serial correlation of residuals is detected (in a Box-Pierce χ^2 test, not reported here).

Similar patterns emerge in the VAR(2) case. In particular, we can explain 8.99 percent of the variations in consumption growth and 76.1 percent of the variations in inflation. We find that the two lags of inflation are significant in explaining both inflation and consumption growth, while the two lags of consumption growth enter significantly in the inflation regression only. An asymptotic $\chi^2(4)$ test of exclusion of the cross-dependence coefficients $(a_{12}^{(1)}, a_{12}^{(2)}, a_{21}^{(1)}, a_{21}^{(2)})$, where $a_{ij}^{(k)}$ is the (i, j) component of the (k)th coefficient matrix $\mathbf{A}_k$, yields a test statistic of 20.136 with a P-value of .0047. These results for the VAR(2) conform with the result for the VAR(1) regarding the statistical significance of the addition of cross-terms. We also find that the test statistic for inflation's significance in the consumption growth regression $(\chi^2(a_{12}^{(1)}, a_{12}^{(2)}) = 2.928(.231))$ is insignificant, while the test statistic for consumption's significance in the inflation regression $(\chi^2 a_{21}^{(1)}, a_{21}^{(2)} = 18.619(.0000))$ is highly significant.

Figure 1 plots price levels and per capita consumption in log-growth terms.The negative relation between the series is most obvious since the late sixties, and to a lesser extent in the late fifties. Figures 2 and 3 are of the observed, fitted, and residuals series from the VAR(1) regression, starting 1953Q1. Consumption is not highly predictable and seems homoskedastic. Inflation is highly predictable and heteroskedastic. Patterns in the residual series will be discussed later.

In summary, we see that most of the variation is captured well by a VAR(1). In addition, there are statistically significant cross-dependencies between the series. These conclusions are robust to the inclusion of durable goods in the consumption series, to changes in the sampling frequency (for example, semiannual and annual), and are stable over various subperiod.

Conditional Heteroskedasticity

The innovations resulting from the VAR exhibit special patterns of autoregressive conditional heteroskedasticity. Table 1 documents statistical analysis of heteroskedasticity. Three different tests for heteroskedasticity are conducted. HET1 is Engle's (1982) LM test for ARCH(1) (distributed asymptotically $\chi^2(1)$). HET2 is the Breusch-Pagan test with squared residuals regressed on lagged consumption growth, lagged inflation and lagged squared residuals. The test statistics are distributed asymptotically $\chi^2(3)$ for a VAR(1) and $\chi^2(5)$ for a VAR(2). HET3 is the

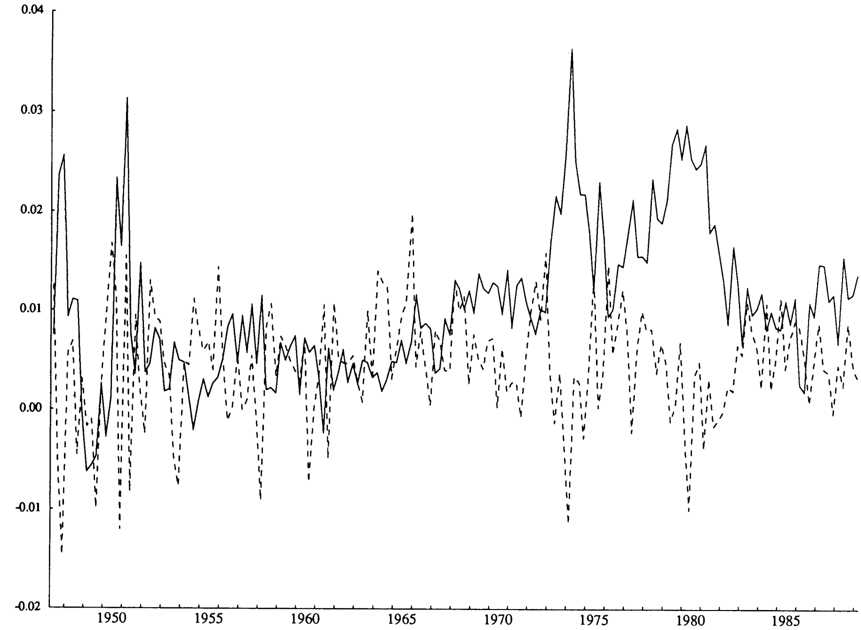

FIG. 1. Log Consumption Growth and Log Inflation
Consumption (dashed line) and inflation (solid line) are calculated using National Income and Product Accounts data for the period 1947Q1 to 1989Q1. Calculations are based on per capita consumption data of nondurables and services and the implicit price deflator.

Breusch-Pagan test *without* the lagged squared residuals, distributed asymptotically $\chi^2(2)$ for VAR(1) and $\chi^2(4)$ for VAR(2). HET1 is sensitive to ARCH-type deviations from homoskedasticity, HET3 is sensitive to level-dependent heteroskedasticity, and HET2 is sensitive to both.

No evidence of conditional heteroskedasticity is found in the consumption series, for both the VAR(1) and the VAR(2), for all three tests. The VAR(1) residuals of inflation uncover an interesting pattern. While the ARCH LM test statistic borders rejection of homoskedasticity, with HET1 = 3.332 (p-value = .0679),the Breusch-Pagan tests are insignificant: HET2 = 5.388(.145) and HET3 = 2.871(.238). The difference between HET2 and HET3 shows that the Breusch-Pagan test statistic is sensitive to the omission of lagged squared errors, in line with the relatively high HET1 test statistic. Put together, these tests supply evidence indicative of autoregressive conditional heteroskedasticity in the inflation time series, which cannot be related to lagged levels of the series, but can be related to its own lagged volatility.

Consistent with these results and results we obtain later, Figure 3 shows that innovations to inflation seem to have an autoregressive variance, not necessarily related to inflation's level. For example, shocks seem highly volatile in the mid-fifties, when inflation is low relative to the late seventies, when inflation is high but not as

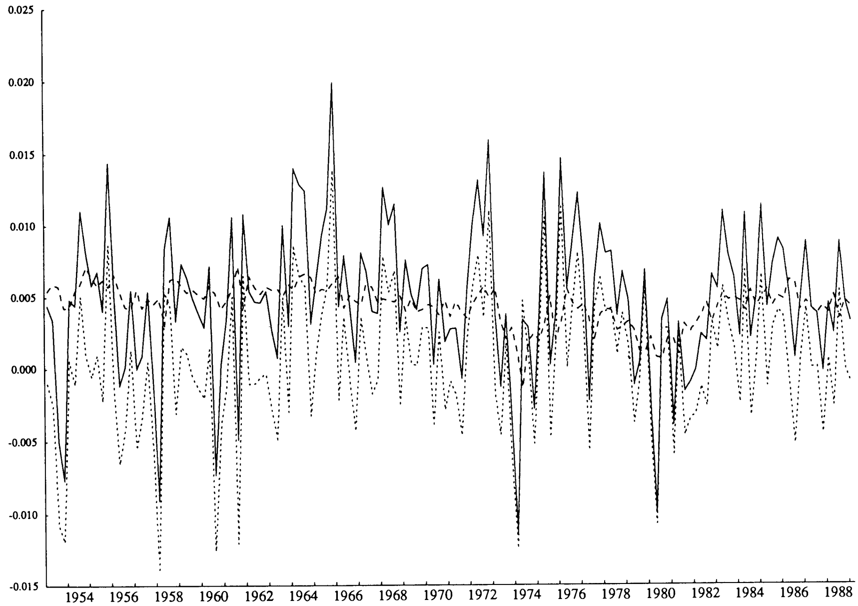

FIG. 2. Log Consumption Growth, VAR(1) Fitted Values and Residuals
Consumption (solid line), VAR(1) fitted values (dashed line), and residuals (dotted line). Data: quarterly per capita consumption of nondurables and services and the implicit price deflator. Source: NIPA. Period: 1953Q1–1989Q1.

volatile. This is not the rule nevertheless: a positive relation between inflation's level and its volatility does exist, in general.

To make this notion more precise (statistically), we specify a level-dependent heteroskedasticity model, estimated in one step with the VAR(1) coefficients. We are interested in the null hypothesis that volatility is, in fact, level independent. Consider the VAR(1) specification for inflation and consumption growth with the following specification for conditional volatility:

$$E_t[\epsilon_{t+1}^2] = \exp(2\Lambda + 2\Omega \mathbf{x}_t) \ ,$$

where Ω and Λ are matrix coefficients.[8] Estimates using the generalized method of moments (GMM) appear in Table 2. In spite of evidence of conditional heteroskedasticity, presented above, the level-dependent volatility specification here does not provide much insight regarding inflation volatility. In particular, a joint exclusion test on $\Omega_{11}, \Omega_{12}, \Omega_{21}, \Omega_{22}$, does not reject. Put together, there is ample evidence to specify the conditional heteroskedasticity as it is presented in the VAR-SV model: Consumption growth is homoskedastic, while inflation is heteroskedastic, with autoregressive (but not necessarily level-dependent) conditional heteroskedasticity.

8. An alternative specification, where $E_t[\varepsilon_{t+1}^2] = \Lambda + \Omega \mathbf{x}_t)^2$, yields qualitatively similar results.

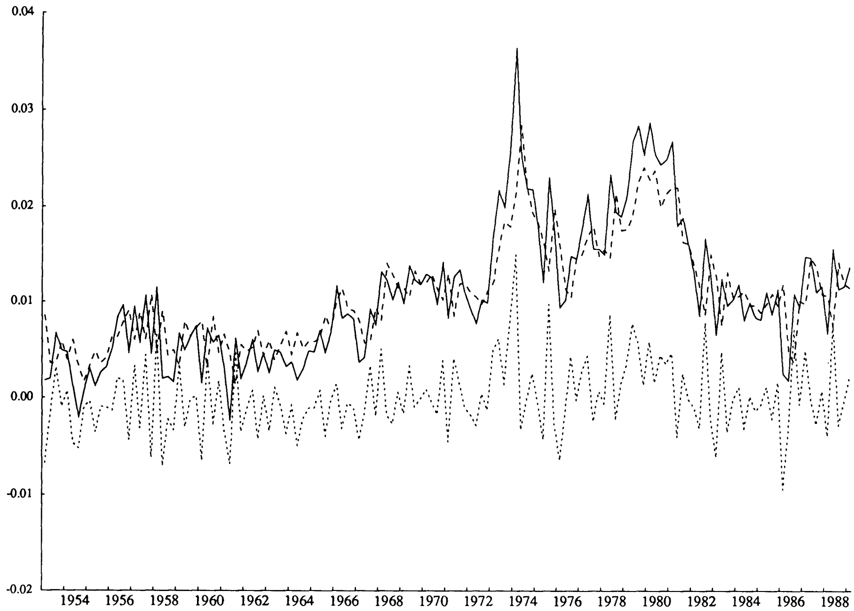

FIG. 3. Log Inflation, VAR(1) Fitted Values and Residuals
Inflation (solid line), VAR(1) fitted values (dashed line), and residuals (dotted line). Data: quarterly per capita consumption of nondurables and services and the implicit price deflator. Source: NIPA. Period: 1953Q1–1989Q1.

TABLE 2

LEVEL-DEPENDENT VOLATILITY SPECIFICATIONS

Parameter	Estimate	(s.e.)
b_1	.005	(.001)
a_{11}	.176	(.089)
a_{12}	−.098	(.0685)
b_2	.14e-3	(.75e-3)
a_{21}	.246	(.066)
a_{22}	.881	(.049)
ρ_{gi}	−.379	(.081)
Λ_1	−5.196	(.145)
Ω_{11}	−9.635	(13.037)
Ω_{12}	−9.146	(9.276)
Λ_2	−5.622	(.111)
Ω_{21}	−2.327	(11.414)
Ω_{22}	−.645	(7.659)
Statistic	Value	(P-value)
$\chi^2(\Omega_{11},\Omega_{12})$	1.281	(.527)
$\chi^2(\Omega_{21},\Omega_{22})$	.0418	(.979)
$\chi^2(\Omega_{11},\Omega_{12},\Omega_{21},\Omega_{22})$	1.319	(.858)
$\chi^2(2)$	2.094	(.351)

Data: 1953Q1–1989Q1 per capita consumption of nondurable goods and services and the implicit price deflator. Source: NIPA. GMM estimation of VAR(1) with lagged consumption growth and inflation determining conditional volatility log-linearly. Standard errors are in parentheses. The exclusion tests are calculated using the covariance matrix of the parameters obtained via GMM. Instruments are a constant, consumption growth and inflation lagged once. The $\chi^2(2)$ uses the overidentifying restriction.

2c. *Estimation of the VAR-SV Model*

The estimation of the VAR-SV model in (8)–(11) is conducted using GMM. The method and specification are an extension of Melino and Turnbull (1990). We estimate using an overidentified system. The main difficulty arises in the identification of the parameters governing the unobservable autoregressive process for inflation volatility. With respect to consumption growth, we use conditioning data for the first three powers of the innovations, ϵ^{g}_{t+1}. Odd moments are conditionally zero, while the second moment's expectation is time invariant: σ^2_g. We also impose the moment restriction that the first-order serial correlation of these innovations is zero, under the null of the VAR-SV model.

Since an unobservable process governs inflation volatility, we use analytic expressions for various unconditional moments of functions of the errors and their lags. For example,

$$E\left[\,|\,\epsilon^{i}_{t+1}\,| - \sqrt{2/\pi}\,\exp(M_v + .5V_v)\right] = 0$$

where $M_v = b_3/(1 - a_{33})$ and $V_v = \sigma^2_v/(1 - a^2_{33})$. Here M_v is the unconditional mean and V_v is the unconditional variance of the process governing the logarithm of inflation volatility. The moments used for estimation are detailed in Appendix A.

Table 3 contains the results of the estimation. The parameters related to the VAR are roughly similar to those in Table 1. The discrepancy is due to the use of additional orthogonality conditions over and above those implied by the normal equations under OLS. Of more interest here, the estimated autoregression parameter of inflation volatility (a_{33} = .829) implies a half- life of 3.7 periods for the shocks (approximately one year). We relate the high standard error to the difficulty in pinpointing

TABLE 3

THE VAR-SV MODEL ESTIMATION

Parameter	Estimate	(s.e.)
b_1	.006	(.97e-3)
a_{11}	.199	(.0784)
a_{12}	−.154	(.0600)
b_2	.11e-4	(.58e-3)
a_{21}	.213	(.0577)
a_{22}	.892	(.038)
ρ_{gi}	−.289	(.103)
σ_g	.004	(.30e-3)
b_3	−1.029	(2.501)
a_{33}	.829	(.415)
σ_v	.258	(.280)
ρ_{iv}	.716	(.449)
Statistic	**Value**	**(P-value)**
$\chi^2(a_{33},\sigma_v)$	2702.8	(.000)
$\chi^2(a_{12},a_{21})$	16.73	(.000)
$\chi^2(13)$	66.25	(.000)

Data: 1953Q1–1989Q1 per capita consumption of nondurable goods and services and the implicit price deflator. Source: NIPA. Estimation of VAR-SV model in (8)–(11). Standard errors are in parentheses. The exclusion tests are calculated using the covariance matrix of coefficients from the second stage of GMM. Instruments are a constant and consumption growth and inflation lagged once. The $\chi^2(13)$ is using the overidentifying restrictions.

the parameters of the unobservable process. The correlation between shocks to inflation and inflation volatility is estimated to be $\rho_{iv} = .716$, demonstrating the positive link between these innovations. Hence, a positive link exists between unexpected positive shocks to inflation and unexpected positive shocks to inflation volatility, and ex post, between levels of inflation and inflation volatility. This positive link between level and volatility is typical to a number of time series models which attempt to capture heteroskedasticity. For example, the inflation model in CIR implies $\text{var}(y) = k^2y$, where y is the expected inflation level and k^2 is related to the speed of reversion. Unlike the inflation model in CIR, the link between inflation volatility and its expected level is, however, potentially imperfect.

Two additional observations are in place. First, the correlation of the shocks to inflation and consumption growth $\rho_{gi} = -.2886$, is significantly negative, in line with previous results in this section. Second, the eigenvalues of A are 4.006 and 1.189, both outside the unit circle, providing some assurance of stationarity.

3. DETERMINANTS OF THE TERM STRUCTURE

In this section we study the properties of the term structure of interest rates, implied by the time series properties of fundamentals. We focus on the economic importance of the statistical properties special to the VAR-SV model. After a brief discussion of the approximation methods we use to price bonds, we calibrate the VAR-SV model's implied bond prices (given the parameters in Table 4) with historical data. We then investigate the roles of consumption, inflation, and inflation volatility in determining real and nominal bond prices and bond returns.

3a. The Approximation Technique

The pricing model (4) in conjunction with the VAR-SV model do not lend themselves to closed-form solutions for bond prices,with the exception of the spot rate, due to the deviation from log-linearity. As such, we study the results of a numerical approximation technique. We extend the *Gaussian quadrature method* for the case of stochastic volatility. Here we include a brief description of the method [see Boudoukh (1991) for complete details].

TABLE 4

CALIBRATION OF THE VAR-SV MODEL WITH BOND DATA

Variables	Moments	β (s.e.)	γ (s.e.)	$\chi^2(\text{df})$ = Val (P-value)
3m,1y	Set 1	.966 (.002)	−2.481 (.217)	$\chi^2(2)$ = 18.531 (.000)
and 5y	Set 1	1.017 (.003)	2.964 (.280)	$\chi^2(4)$ = 58.801 (.000)
All	Set 1	1.015 (.002)	2.790 (.173)	$\chi^2(12)$ = 88.353 (.000)
3m,1y	Set 2	1.004 (.002)	4.958 (.182)	$\chi^2(4)$ = 21.199 (.000)
and 5y	Set 2	1.019 (.002)	3.252 (.198)	$\chi^2(7)$ = 60.110 (.000)
All	Set 2	1.022 (.001)	3.584 (.1118)	$\chi^2(19)$ = 94.413 (.000)

Data: 1959Q1–1985Q1 bond yields, for three and six months to maturity, and one, two, three, four, five years to maturity. Source: CRSP. Parameters: The VAR-SV model parameters are from Table 3. β and γ are the agent's intertemporal substitution and relative-risk-aversion parameters. Standard erros are in parentheses. Set 1 consists of means and variances of yields; set 2 adds autocovariance of yields to Set 1. All moments are unconditional. The $\chi^2(j)$ is the specification test using the overidentifying restrictions.

The method relies on discretizing the continuous state space, in which the state variables evolve. The state is approximated by a finite number of discrete points. For example, for the one-period bond:

$$Q_t^1 = \int\int\int \beta G_{t+1}{}^{-\gamma} I_{t+1}{}^{-1} f(G_{t+1}, I_{t+1}, V_{t+1} \mid G_t, I_t, V_t) dG_{t+1} dI_{t+1} dV_{t+1}$$
$$\approx \Sigma_1^N \beta \hat{G}_{t+1}^{-\gamma} \hat{I}_{t+1}^{-1} \Pi(\hat{G}_{t+1}, \hat{I}_{t+1}, \hat{V}_{t+1} \mid \hat{G}_t, \hat{I}_t, \hat{V}_t)$$

where $N = N_G \cdot N_I \cdot N_V$, ($N_j$ is the number of discretization points for state variable j). The processes $\hat{G}, \hat{I}, \hat{V}$ with the transition matrix $\Pi(\cdot|\cdot)$ are the discrete state space counterparts of G, I, V with the transition matrix $f(\cdot|\cdot)$. The approximating Markov chain is determined by a Gaussian quadrature rule which matches moments $1, 2, \ldots, N$ in the continuous state space to their discrete state space counterparts, where N is the number of discretization points. For example, in the univariate case, for $N = 3$ quadrature points the approximating three-state Markov chain (the states and the transition probabilities) is chosen so that the first three moments match exactly those of the continuous state space process. Tauchen and Hussey (1991) implement the method for equity pricing, and document some accuracy results. Boudoukh and Whitelaw (1989) extend the methods to the pricing of a class of interest rate dependent securities (for example, American options on bonds and mortgagebacked securities).

3b. Calibration

Using the parameters obtained for the VAR-SV model and using historical data on bond prices, we estimate the preference parameters in (4), namely β and γ. The data are taken from CRSP, and correspond to the data in Fama (1984) and Fama and Bliss (1987). The data are sampled quarterly from yields on three-, six-, twelve-, twenty-four-, thirty-six-, forty-eight-, and sixty-months-to-maturity bonds. We use GMM to calibrate the model by matching unconditional moments of bond yield data to their theoretical counterparts. Theoretical bond prices are obtained for the estimated law of motion (using the parameters in Table 4), and different sets of preference parameters (γ, β) over which the search occurs.

The results, documented in Table 4, present the estimates and standard deviations of β and γ, as well as a specification test statistic for a number of combinations of data and moments. The moments considered are the unconditional means and variances of yields (Set 1) and Set 1 plus the autocorrelation of yields (Set 2). The estimates of β and γ range from .966 to 1.022 and from -2.481 to 4.958 respectively.

The estimation is done using means, variances, and autocovariances of yields of various maturity bonds. We notice that the specification test statistic is high in all cases. This is not surprising in light of previous evidence in the literature of rejections of the representative consumer specification (for example, Hansen and Singleton 1983, Grossman, Melino, and Shiller 1987, Hansen and Jagannathan 1991). The specification test statistics are valid only if we ignore the fact that the inputs are

TABLE 5

COMPARISON OF SUMMARY STATISTICS: THE DATA AND THE MODEL

	3m	1Y	2Y	3Y	4Y	5Y
5.1: Fama Bliss Data 1959Q1–1985Q1						
Mean	.0629	.0650	.0663	.0674	.0682	.0685
Std.Dev.	.0307	.0277	.0277	.0271	.0271	.0269
Autocor.	.9036	.9182	.9380	.9484	.9555	.9610
5.2: The VAR-SV Model						
Mean	.0662	.0661	.0661	.0660	.0660	.0659
Std.Dev.	.0217	.0190	.0170	.0154	.0139	.0127
Autocor.	.8488	.9398	.9453	.9455	.9454	.9452
5.3: The Log-Linear Model						
Mean	.0662	.0659	.0658	.0658	.0657	.0657
Std.Dev.	.0199	.0103	.0068	.0050	.0039	.0032
Autocor.	.4237	.6686	.7318	.7462	.7509	.7527

Table 5.1 lists the means, variances, and autocorrelations of bond yields using the data from the Fama files on CRSP. Table 5.2 lists similar statistics for the VAR-SV model's *implied* yields, using the parameters in Table 3. Preference parameters are from the last line in Table 4. The statistics in Table 5.3 are obtained using the log-linear model and parameters from Table 1 (the VAR(1) case). Preference parameters are similar to Table 5.2 except $\beta = 1.001$ to match the implied mean spot rate across models.

a result of a first-step estimation of the VAR-SV parameters, and will be used therefore only as a tool for comparison in the calibration.

Ideally, one would like to estimate the entire model in one step. Attempts to do so failed due to the size of the parameter space, and the unobservability of the state of inflation volatility. This unobservability limits our ability to use conditioning information, thus making the identification of parameters a difficult task. The parameter estimates for γ and β remain useful for our purposes, since they still represent the best pair of parameters available when matching the VAR-SV model with bond prices. The dimensions along which we fail to match bond prices are discussed later.

The preference parameters which we choose to use in the analyses below are taken from the last row in Table 4, which employs all the data and Set 2 of the moments. The parameters are $\gamma = 3.584$ and $\beta = 1.022$. The value for the relative risk aversion is "reasonable," while β is high due to the inclusion of long-maturity bonds, which tend to increase the estimate of β. The moments that we use for estimation: means, variances, and first-order autocorrelations, for maturities ranging from three months to five years, are documented in Table 5.

3c. The Level, Volatility, and Autocorrelation of Longer-Term Yields

Bond Yields Assuming Homoskedasticity

Table 5 compares moments of the data to similar moments that are derived using estimated parameters for the VAR-SV model and for the LLM. Table 5.1 lists the means, variances, and autocorrelations of bond yields using the data from the Fama files on CRSP. Table 5.2 lists these moments for the VAR-SV model, which are computed using the parameters in Table 3 and the last line in Table 4. The moments in Table 5.3 are obtained using the LLM and parameters from Table 2 (the VAR(1) case). The relative-risk-aversion parameters used are the same in Tables 5.2 and 5.3, while β is 1.001, in order to match the implied mean spot rate across models.

Consider first the comparison of the moments between Table 5.1 and Table 5.3. The mismatch between the data and the LLM is apparent. Standard deviations and autocorrelations of yields across maturities are higher in the data than those implied by the LLM. In particular, spot rates implied by the LLM are 35 percent less variable (with a standard deviation of .0199 versus .0307) and 53 percent less autocorrelated (.9036 versus .4237). At the long end of the term structure the variability of yields is 88 percent lower in the LLM, while the autocorrelation is 22 percent lower.

This discrepancy is often attributed to the low variability of consumption data, the time separability of preferences, the sampling frequency, etc. We elaborate here on the effect of simplifying assumptions, often made for convenience reasons, with respect to the time series properties of the fundamental processes.

In the VAR-SV model (Table 5.2) we respecify the structure of the uncertainty in the economy, and find that the implied prices of bonds are more in line with the data. The autocorrelations of yields are higher, in line with the data, and the rate of decay of yield volatility of longer bonds is lower. In particular, spot rates implied by the VAR-SV are 27 percent less variable (with a standard deviation of .0217 versus .0307) and only 6 percent less autocorrelated (.9036 versus .8488). At the long end of the term structure the variability of yields is 53 percent lower in the VAR-SV, while the autocorrelation is only 2 percent lower. The decay in variability in the data, quantified by the ratio of the five-year bond yield's standard deviation to that of the three-month bill, is $s.d.(5y)/s.d.(3m) = .02687/.03066 = .876$. The ratio implied by the LLM is much lower, namely .159, and the VAR-SV model shows a marked improvement, with a ratio of .586.

These improvements can only be the result of the respecification of the time series model for inflation and consumption, since the pricing model and the data are common to the results in Tables 5.2 and 5.3. Deviations from the expectations hypothesis and some of the joint properties of long and short rates [for example, the overreaction phenomenon in Shiller, Campbell, and Schoenholtz (1983)] are hence captured, at least to some extent, by the stochastic volatility model.

Bond Yields Assuming Consumption-Inflation Independence

Above we saw that the VAR-SV model matches the data better than the LLM, due to its generality in modeling time variations in volatility. Both models, however, allow for consumption-inflation dependence, therefore relaxing the commonly used assumption that inflation is orthogonal to consumption. What is the economic role of relaxing this assumption? To answer the question, we reestimate the VAR-SV model assuming independence. We assume that $a_{21} = a_{12} = \rho_{gi} = 0$, and call the restricted model the R-VAR-SV model. While there is ample statistical evidence against this restriction, it is incumbent upon us to investigate the economic importance of inflation-output correlation in the determination of interest rates.

Table 6.1 presents results of an estimation procedure that is similar to the one

TABLE 6

THE RESTRICTED VAR-SV MODEL ESTIMATION AND CALIBRATION

6.1: The R-VAR-SV Model Estimation

Parameter	Estimate	(s.e.)
a_{11}	.004	(.62e-3)
b_1	.254	(.085)
σ_g	.005	(.25e-3)
b_2	.002	(.36e-3)
a_{22}	.844	(.039)
b_3	−.924	(3.021)
a_{33}	.850	(.490)
σ_v	.263	(.388)
ρ_{iv}	.581	(.608)

6.2: Calibration of the R-VAR-SV model with Bond Data

Variables	Moments	β (s.e.)	γ (s.e.)	χ^2(df) = Val (P-value)
All	Set 2	.997 (.0012)	1.704 (.241)	$\chi^2(12)$ = 90.45(.000)

6.3: Yield Moments

	3m	1Y	2Y	3Y	4Y	5Y
Mean	.0711	.0710	.0709	.0709	.0709	.0708
Std.Dev.	.0141	.0102	.0086	.0075	.0066	.0058
Autocor.	.6569	.8723	.9086	.9164	.9193	.9206

Data: 1953Q1–1989Q1 per capita consumption of nondurable goods and services and the implicit price deflator. Source: NIPA. Estimation of the VAR-SV model in (8)–(11), assuming $a_{12} = a_{21} = \rho_{gi} = 0$. Standard errors in parentheses.

used for the unrestricted VAR-SV model. We specify an autoregressive process with stochastic volatility for inflation, and an autoregressive process for consumption growth. Since we assume independence, the parameters driving consumption growth are simply the AR(1) parameters. The estimate of the standard deviation of shocks to consumption growth is $\sigma_g = .005$. Specifying an AR-SV process for inflation, we need to estimate six parameters: $b_2, a_{22}, b_3, a_{33}, \sigma_v, \rho_{iv}$. The autoregressive parameter of inflation is $a_{22} = .844$ (the AR(1) estimate of this parameter in Table 1 is .838). Inflation volatility is autoregressive: $a_{33} = .850$, and close to the unrestricted VAR-SV estimate, .829, in Table 3.

The next step, calibration, yields parameter estimates for the preference parameters β and γ, that are quite different from their counterparts in the unrestricted VAR-SV estimation. Table 6.2 reports only results that are comparable to line six of Table 4, in which all the data (three months up to five years-to-maturity bond yields) are used, as well as the full set of moments: means, variances, and autocorrelations. The estimate for the relative risk aversion parameter is $\gamma = 1.704$ and the discount rate estimate is $\beta = .997$.

Moments of yields implied by the set of parameters we estimated appear in Table 6.3. These are comparable to the moments reported in Tables 5.2 and 5.3 for the unrestricted VAR-SV.Comparison of the empirical moments (Table 5.1) with those of the VAR-SV (Table 5.2) and the restricted VAR-SV (Table 6.3) shows an improvement (in terms of matching moments) in the VAR-SV over the restricted VAR-SV in a number of meaningful ways. In particular, the VAR-SV performs better in matching average yields, the rate of decay of volatility is smaller ($s.d.(5y)/s.d.(3m)$ is

.876 in the data, .586 in the VAR-SV model, and .413 in the restricted VAR-SV), and the autocorrelation is higher.

3d. Real Rates and Inflation

The continuously compounded nominal and real spot rates can be written in terms of the state variables:

$$\begin{aligned} R_t^1 &= \ln(\$1/Q_t^1) \\ &= -\ln\beta + \gamma b_1 + b_2 + (\gamma a_{11} + a_{21})g_t + (\gamma a_{12} + a_{22})i_t \\ &\quad - .5\gamma^2\sigma_g^2 - .5V_t^2 - \gamma\rho_{gi}\sigma_g V_t\,, \end{aligned} \tag{12}$$

$$\begin{aligned} r_t^1 &= \ln(1/q_t^1) \\ &= -\ln\beta + \gamma b_1 + \gamma a_{11}g_t + \gamma a_{12}i_t - .5\gamma^2\sigma_g^2\,. \end{aligned} \tag{13}$$

Consumption growth and inflation enter linearly, and inflation volatility enters both linearly and quadratically to determine nominal rates.

Using the parameter estimates (Tables 3 and 4) we can assess the marginal effect of the state variables on the nominal and real spot rate. Spot rates are increasing in current consumption growth ($\partial R_t^1/\partial g_t = \gamma a_{11} + a_{21} = .913$) and inflation ($\partial R_t^1/\partial i_t = \gamma a_{12} + a_{22} = .350$). The strong autocorrelation of inflation results in two effects. One is a large positive effect on nominal rates ($a_{22} = .892$). The other is a mitigating effect (a negative effect) inflation has indirectly on consumption growth (since $a_{12} = -.154$). This effect is inflated by the relative-risk-aversion parameter ($\gamma = 3.581$). Real spot rates are increasing in consumption growth ($\partial r_t^1/\partial g_t = \gamma a_{11} = .700$), an effect of the positive autocorrelation of consumption growth, inflated by the relative-risk-aversion parameter, and decreasing in inflation ($\partial r_t^1/\partial i_t = \gamma a_{12} = -.542$).

We separate the nominal rate according to (13) into a constant plus three state-related terms. The state independent term, namely ($-\ln\beta + \gamma b_1 + b_2 - .5\gamma^2\sigma_g^2$), accounts for 36.5 percent of the spot rate. The first two components, which are related to consumption growth, account for 52–64 percent of nominal rates when consumption growth is high. In these states inflation-related components, namely $\gamma a_{12} + a_{22}$, contribute together 7–23 percent. Inflation's contribution consists of two opposite effects. The first component, $\gamma a_{12}i_t = -.54i_t$, the indirect effect of inflation via its effect on consumption growth, enters nominal rates (and real rates) negatively in every state. It accounts, on average, for -22.1 percent of the nominal rate. The second component, $a_{22}i_t = .83i_t$, the direct effect of inflation on future inflation, enters nominal rates (*but not* real rates) positively in every state. It accounts, on average, for 36.3 percent of the nominal rate.

Inflation, therefore, results in two effects: it tends to directly decrease nominal bond prices, but also to indirectly increase them. This happens because it tends to decrease consumption growth, which increases nominal (and real) bond prices. The second effect overwhelms in magnitude the first effect, due to the high persistence in inflation (as reflected in the a_{22} parameter), so long as the risk aversion parameter is low.

Nominal spot rates are relatively unaffected by inflation volatility ($\partial R_t^1/\partial(\exp(v_t)) = \exp(v_t) + \gamma\rho_{gi}\sigma_g$, which varies with the state between 0 and $-.003$). Real spot rates are not affected by inflation volatility (although rates that are two-period rates and beyond are affected indirectly via the levels and volatility of the other state variables).

3e. The Relative Variability of Real Rates and Inflation

What are the implications of these estimates for the variability of real rates versus inflation? It is easier here to work with real bond prices to get closed forms. Using standard results regarding the variance of lognormal processes we get

$$\text{var}_t[q_t^1] = \exp(-2\gamma(b_1 + a_{11}g_t + a_{12}i_t) + \gamma^2\sigma_g^2) \cdot [\exp(\gamma^2\sigma_g^2) - 1] \qquad (14)$$

$$\text{var}_t[\exp(-i_{t+1})] = \exp(-2(b_2 + a_{21}g_t + a_{22}i_t) + V_t^2)[\exp(V_t^2) - 1]\,. \qquad (15)$$

Since q_t^1 is close to one, its variance would be a good approximation for that of r_t^1. We find that real rates are 2.4 to 6 times more volatile (in standard deviation terms) than inflation, depending on the state. The result is surprising considering the conclusions of Fama (1975) and Mishkin (1981), which attribute most of the variation in the term structure to inflation, but consistent with results in recent work of Pennacchi (1991).

To reconcile the two seemingly opposing conclusion, notice that it is possible for a random variable to have a higher average conditional variance then another random variable, but lower unconditional variance. Considering the familiar decomposition of the unconditional variance: $\text{var}[x] = E[\text{var}_t[x]] + \text{var}[E_t[x]]$ we identify the conclusions here and in Pennacchi as being with regard to the average conditional variance term, which is found to be higher for real rates than for inflation. Other studies conclude that the variability of the conditional expectations of inflation are higher than those for real rates. These two are not inconsistent with each other or with the fact that the overall variance of inflation is higher than that of real rates. These conclusions are also consistent with the fact that inflation is more predictable and slower to mean revert than real rates. Therefore, of its total variability, more (in relative terms) is due to variations in the conditional mean, and less is due to variations about the conditional mean.

For a univariate AR(1) (that is, $x_{t+1} = b + ax_t + \epsilon_{t+1}$) for example, it is easy to show that $E[V_t[x_{t+1}]] < V[E_t[x_{t+1}]]$ if and only if $a > \sqrt{1/2}$. Since inflation is highly persistent, an inequality as above holds. An inequality of the opposite sign holds for real rates. A simple exposition of the result is achieved by estimation of a univariate autoregression for inflation and real rates using interest rate and inflation data. In the regression of inflation $a = .85$ and $\sigma = .016$ (the standard deviation of the error term). Thus,

$$V[i_{t+1}] = E[V_t[i_{t+1}]] + V[E_t[i_{t+1}]] = .015^2 + 2.5 * .015^2 = 5.6e - 4 ,$$

where $2.5 = a^2/(1 - a^2)$. For real rates $a = .73$ and $\sigma = .018$, thus,

$$V[r_{t+1}] = E[V_t[r_{t+1}]] + V[E_t[r_{t+1}]] = .018^2 + 1.1 * .018^2 = 3.7e - 4 .$$

This simple calculation shows that indeed the variability of expected inflation is 50 percent higher than that of real rates, the average conditional variability of real rates is 45 percent higher than that of inflation, and that altogether, the unconditional variance of inflation is 13 percent higher than that of real rates.

3f. Implied Volatility

The state of inflation volatility is not directly observable. It is possible, however, to solve for an "implied volatility," using the observable state variables, the spot rate, and the estimated parameters. The price of a one-period bond, given inflation and consumption growth, is

$$\begin{aligned} R_t^1 &= \ln \frac{\$1}{Q_t^1} \\ &= -\ln(\beta) + \gamma b_1 + b_2 + (\gamma a_{11} + a_{21})g_t + (\gamma a_{12} + a_{22})i_t \\ &\quad - .5\gamma^2\sigma_g^2 - .5V_t^2 - \gamma\rho_{gi}\sigma_g V_t , \end{aligned} \tag{16}$$

which is a quadratic in V_t. Solving with respect to V_t and taking only the positive root, we get

$$V_t = -\gamma\rho_{gi}\sigma_g + \sqrt{(\gamma\rho_{gi}\sigma_g)^2 - 2(\lambda_t + .5\gamma^2\sigma_g^2)} , \tag{17}$$

where

$$\lambda_t \equiv R_t^1 - (-\ln(\beta) + \gamma b_1 - b_2 + (\gamma a_{11} + a_{21})g_t + (\gamma a_{12} + a_{22})i_t) .$$

Solutions do not always exist. In Figure 4 we document the time path of the implied volatility. The top figure is for the parameter set that was estimated in Tables 3 and 4. When the solution does not exist, we simply observe that the line is flat (due to the arbitrary assignment of zero to the square root term). The bottom figure uses a higher relative-risk-aversion parameter, arbitrarily chosen as $\gamma = 10$, and we observe that there are now many fewer cases in which the implied volatility does not exist.

The only two periods in which the implied volatility does not exist (with $\gamma = 10$) are during the 1973Q4–1974Q4 period of the oil crisis, and during the 1979Q4–1982Q4 period, coinciding with the change in the Federal Reserve operating procedures. Hamilton (1988), for example, characterizes the 1979Q4–1982Q4 period as a period in which a "different regime" prevailed. His methodology is designed to

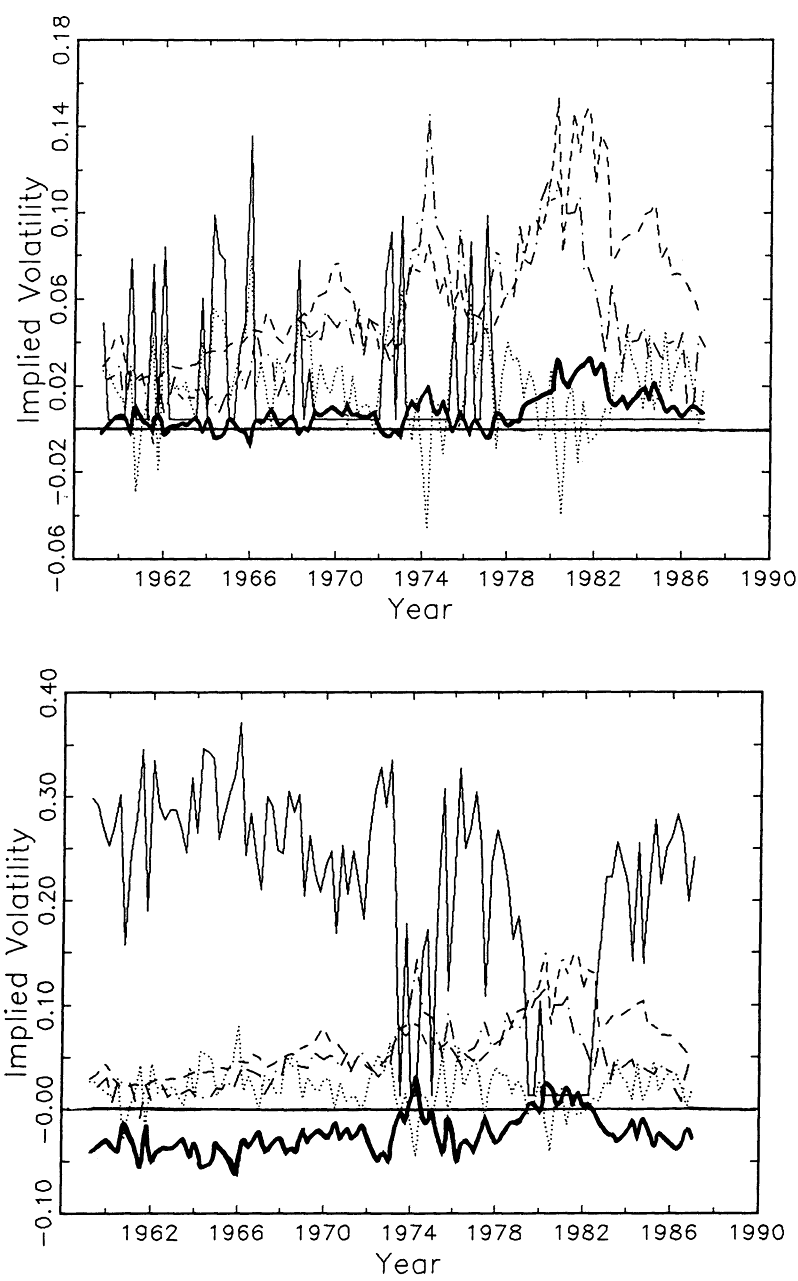

FIG. 4. Implied Volatility

Implied volatility (solid line), spot rate (dashed line), consumption growth (dotted line), inflation (dashed-dotted line), and $-.5\gamma^2\sigma_g^2 - .5V_t^2 - \gamma\rho_{gi}\sigma_g V_t$ (thick solid line). Implied volatility, when it exists, is the positive root of the quadratic equation in (16), using the current state of interest rates, inflation, and consumption growth, and using parameter estimates from Tables 3 and 4. The top figure is for $\gamma = 3.584$, and the bottom figure is for $\gamma = 10$. When the implied volatility is at the minimum, a real solution does not exist.

capture statistically a regime switch.The probability that market participants assign to the possibility of a different regime prevailing in a given quarter is identified via maximum likelihood. Hamilton estimates a high probability of such a regime shift during the 1979Q4–1982Q4 period. Hamilton also identifies an eight-fold increase in the probability assigned to a regime shift (relative to the previous six-period average probability of a switch), in 1973Q3. The estimated probability is, nevertheless, rather low. The probability is .0196 relative to an average probability of .0024 during 1972Q1–1973Q2. This effect could be attributed to the highly restrictive nature of the assumption, made a priori, that there are only two regimes possible. If, in fact, during the 1973–1974 period a third "regime" prevailed, it might show up as a small probability of a regime shift. The VAR-SV can be viewed as more flexible from this perspective.

3g. The Risk Premium in the Term Structure

As discussed above, when the covariance matrix is time invariant, as in the LLM, the only wedge between the forward rate and the expected spot rate is a time-invariant premium. Because the LLM is nested within the VAR-SV model, variations in inflation's volatility are the only source of risk and risk premium *variations*. Can the unobservable factor governing the volatility explain the time series properties of the risk premium? In particular, is it the case that the parameters estimated for the VAR-SV model imply similar means, variances and autocorrelations to those obtained from bond yields?

Backus, Gregory, and Zin (1989) analyze the time series properties of spot and forward rates and of the risk premium under (4). They discretize the nominal marginal rate of substitution and show that risk premia, which are a function of the autocovariance of the marginal rate of substitution,[9] can account for neither the sign nor the magnitude in bond data. Their economy is also incapable of generating enough variation in risk premium. Our findings here agree with the former conclusion but not with the latter. This difference is due to the more elaborate analysis undertaken here, of variations of the nominal marginal rate of substitution, and the analysis of its components: the real marginal rate of substitution and inflation. The theoretical risk premium between the three- and the six-month rate is shown to be approximately as variable as that in the data. It is also shown, however, that the puzzle regarding the observed high mean of the premium is preserved.

Table 7 compares the data to the theoretical model, where $F_t(t+1, t+2) = Q_t^2/Q_t^1$ is the forward price and $\Phi_t(t+1, t+2) = E_t[Q_{t+1}^1 - F_t(t+1, t+2)]$ is the forward premium. The parameters used to derive the theoretical prices are from Tables 3 and 4. Conditional expectations in the theoretical model are known. The risk premium $\Phi_t(t+1, t+2)$ in the theoretical model is close to zero. This finding is consistent with the observations regarding the smoothness of consumption data. The variability of the risk premium is, nevertheless, close to that of the data. As Backus, Gregory, and Zin (1989) note, the variability of the realized risk premium, namely .00357, is the sum of the variability of the conditional risk premium plus the vari-

9. See Shiller (1990) for a complete discussion.

TABLE 7

OBSERVED AND THEORETICAL BOND PRICES AND THE RISK PREMIUM

Bond Prices, 1959Q2–1986Q2	Q_t^1	Q_t^2	$F_t(t+1,t+2)$	$\Phi_t(t+1,t+2)$
Mean	.984	.968	.983	.001
	(.002)	(.004)	(.002)	(.000)
Std.Dev.	.008	.015	.008	.0036 > $\sigma(\Phi_t)$ > 0016
	(.001)	(.003)	(.001)	(.001)
Autocor.	.91	.91	.90	−.08
	(.042)	(.045)	(.049)	(.135)
Results from the VAR-SV Model	Q_t^1	Q_t^2	$F_t(t+1,t+2)$	$\Phi_t(t+1,t+2)$
Mean	.978	.956	.978	−.20e-4
Std.Dev.	.005	.010	.005	.0018
Autocor.	.85	.91	.94	.29

The data is taken from CRSP. The theoretical moments rely on estimated parameters in Tables 3 and 4. Standard errors are in parentheses. The table entry .0036 > $\sigma(\Phi_t)$ > 0016 are the lower and upper bounds of the unconditional standard deviation of the risk premium.

ance of the realized prediction errors, which are orthogonal to the time-varying risk premium. This constitutes a lower bound on the variability of the risk premium: .00157.

The variability of the risk premium in the theoretical economy with the estimated VAR-SV parameters and the calibrated preference parameters imply a standard deviation of .0018, close to, and higher than, the lower bound. Therefore, accounting for the cross-correlation patterns of inflation and output, and including a detailed discretization of the state space [with, in our case, $3 \cdot 3 \cdot 3$ state as opposed to two- or three-state discretization in other studies (for example, Mehra and Prescott 1985)] help to recover more precisely higher moments of the marginal rate of substitution. The fact that we can match the variability of the risk premium but not the average magnitude may imply that efforts to supply a more realistic specification for preferences are not the key to generating risk premiums of large magnitudes. Here we find realistic variations, but we fail to match the mean premium.

4. CONCLUDING REMARKS

Motivated by empirical evidence, we specify a vector autoregressive process for inflation and consumption growth, with time-varying volatility. This VAR-SV process allows for co-dependencies of inflation and consumption growth and lets inflation exhibit stochastic shifts in conditional volatility, driven by an unobservable autoregressive process. To price bonds, we assume the existence of a representative agent with power utility function. Given a set of preference-related parameters and the estimated parameters of the VAR-SV model, we can price nominal and real bonds, using the first-order conditions of the representative agent.

We show that

- The VAR-SV model helps capture important properties of the term structure of interest rates.

- The VAR-SV model generates variations in the risk premium on bonds of similar magnitude to those observed in the data, but a negligible average premium.
- Within our framework real rates are more variable than inflation conditionally but not unconditionally.

One important restriction which was not discussed to this point and which may well be important is that the parameters are assumed fixed. We do allow the covariance of innovation to be time varying, and the expectations are time varying as well. If structural changes cause changes in the correlation coefficient of shocks and the VAR coefficients, the same data series may have very different implications about the shape and dynamics of the term structure, and intertemporal variations of the relationship of real rates, inflation, and nominal rates.

Another restriction is related to the assumptions we make regarding the agent's utility function and the fact that only nondurable goods are considered. There are a number of extensions of the analysis along these lines. With respect to preferences, extensions include the introduction of habit persistence and/or time nonseparable preferences. With respect to durable goods, it is necessary to describe the stream of services rendered by the consumption of durable goods. Some evidence of *partial* reconciliation of the equity premium puzzle, and weaker violations of moments of the marginal rates of substitution are presented in Epstein and Zin (1991) and Heaton (1988) respectively. None of these preempt the importance of the extensions offered in this paper.

On a methodological level, the results here do not demonstrate the only application for the approximation methods developed here. The methods do not restrict us to the study of bond prices, and could be applied to equilibrium equity pricing. This methodology could be applied, for example, to the study of market returns in the presence of stochastic volatility of fundamentals.

APPENDIX A

The closed forms of the moment are

$$E[\epsilon^g_{t,t+1} \otimes Z_t] = 0 \tag{18}$$

$$E[\epsilon^i_{t,t+1} \otimes Z_t] = 0 \tag{19}$$

$$E[(\epsilon^g_{t,t+1}{}^2 - \sigma^2_g) \otimes Z_t] = 0 \tag{20}$$

$$E[\epsilon^g_{t,t+1}\epsilon^i_{t,t+1}\rho_{gi}\sigma_g \exp(M_v + .5V_v)] = 0 \tag{21}$$

$$E[\epsilon^g_{t,t+1}{}^3 \otimes Z_t] = 0 \tag{22}$$

$$E\left[|\epsilon^i_{t,t+1}| - \sqrt{2/\pi}\exp(M_v + .5V_v)\right] = 0 \tag{23}$$

$$E[\epsilon^i_{t,t+1}{}^2 - \exp(2M_v + 2V_v)] = 0 \tag{24}$$

$$E\left[|\epsilon^i_{t,t+1}|^3 - \sqrt{6/\pi}\exp(6M_v + 4.5V_v)\right] = 0 \tag{25}$$

$$E\left[\epsilon^i_{t,t+1}{}^4 - 3/\sqrt{2}\exp(4M_v + 8V_v)\right] = 0 \tag{26}$$

$$E[(\epsilon^i_{t,t+1}\epsilon^i_{t-1,t}) \otimes Z_t] = 0 \tag{27}$$

$$E\left[|\epsilon^i_{t,t+1}\epsilon^i_{t-1,t}| - (2/\pi)^{3/2}\exp(2M_v + .5(1 + a_{33})V_v + (-.5\rho^2_{iv}\sigma^2_v) + \rho_{iv}\sigma_v\Phi(\rho_{iv}\sigma_v))\right] = 0 \tag{28}$$

$$E\left[|\epsilon^i_{t,t+1}\epsilon^i_{t-2,t-1}| - (2/\pi)^{3/2}\exp(2M_v + (1 + a^2_{33})V_v + (-.5\rho^2_{iv}\sigma^2_v a^2_{33}) + \rho_{iv}\sigma_v a_{33}\Phi(\rho_{iv}\sigma_v))\right] = 0 \tag{29}$$

$$E[\epsilon^i_{t,t+1}{}^2\epsilon^i_{t-1,t}{}^2 - \exp(2M_v + (1 + a_{33})V_v) * (\rho^2_{iv}\sigma^2_v + 1)] = 0 \tag{30}$$

$$E\left[|\epsilon^i_{t,t+1}|\,\epsilon^i_{t-1,t} - \sqrt{2/\pi}\exp(2M_v + .5(1 + a_{33})V_v + .5\rho^2_{iv}\sigma^2_v)\right] = 0 \tag{31}$$

$$E\left[|\epsilon^i_{t,t+1}|\,\epsilon^i_{t-2,t-1} - \sqrt{2/\pi}\,\rho_{iv}\sigma_v a_{33} \exp\left(2M_v + \frac{\sigma^2_v(2 + a^2_{33} + a^4_{33})}{1 - a^2_{33}} + .5(\rho_{iv}\sigma_v a_{33})^2\right)\right] = 0 \tag{32}$$

where $\Phi(\cdot)$ is the normal cdf and

$$Z_t \equiv [1, g_t, i_t]'$$

$$M_v \equiv \frac{b_3}{1 - a_{33}}$$

$$V_v \equiv \frac{\sigma^2_v}{1 - a^2_{33}} .$$

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