

Risk, Returns, and Values in the Presence of Differential Taxation

by

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Abstract

We consider risk-return and valuation in an economy segmented by differential personal taxation of debt and equity securities. Because equity and debt are differentially taxed, in equilibrium, the risk-free rate for equity and debt securities are different and there exist two SMLs, one for debt securities and one for equity securities, which relate market (post-corporate tax, pre-personal tax) returns to asset risks. We characterize the conditions under which these two SMLs have the same price of risk (albeit with different intercepts), and show the relation of this characterization to a generalized Miller equilibrium. We also characterize the conditions under which the tax effect of leverage is a linear function of the value of the debt (i.e., the standard APV approach to the valuation of levered firms applies). These necessary and sufficient conditions are similar to the necessary and sufficient conditions for the price of risk being the same in the debt and equity markets. The results are derived in a framework consistent with capital market equilibrium with heterogeneous consumers and are not dependent on the functional form of the utility functions.

Our results are interesting not only on a theoretical level but also for the practice of corporate finance. A typical cost of capital calculation takes the time value of money and the market price of risk to be the same for debt and equity despite the fact that these securities are typically differentially taxed. Using our framework, we are able to correctly calculate the cost of capital for debt and equity instruments and unlever betas taking into account differential taxation. We also show how the risk and cost of capital of *all* assets can be measured relative to an *equity* market index.

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Introduction

In this paper we derive risk-return relations for differentially taxed debt and equity securities within an equilibrium model. Surprisingly, there is little agreement on this issue; academic papers and textbooks contain a variety of risk-return formulas, largely, it seems, formulated on an ad-hoc basis.¹ Consider, for example, the formulas to convert estimated betas of stocks and bonds to estimated betas of firms (i.e., formulas to “unlever” betas to arrive at the “asset” betas of firms). Some authors (e.g., Myers and Ruback 1992, Gilson, Hotchkiss, and Ruback 1998) calculate the asset beta simply as the weighted average of the estimated betas of the firm’s debt and equity. Others, such as Kaplan and Ruback (1995), scale down the debt’s beta as well as the weight of debt in the capital structure by one minus the corporate tax rate. (This unlevering formula is also used in textbooks such as Ross, Westerfield, and Jaffe 1997.) These two different procedures clearly result in different risk-adjusted discount rates even though both are supposed to correspond to the Modigliani and Miller (1963) setting where interest is a deductible expense at the corporate level and there are no personal taxes. More important: Independent of the unlevering formula used, the very idea that debt and equity betas can be unlevered to estimate asset betas implicitly assumes that the same price of risk applies to all beta estimates. This is clearly not a trivial assumption when debt and equity are differentially taxed.

A similar confusion is evident in the use of risk-free returns. A few of the open issues are: Should the benchmark risk-free rate of return for equity securities be before tax (as many textbooks suggest) or after tax (as in, for example, Sick 1990 and Ruback 1986)? If the equity risk-free return should be after tax, after *what* tax? Should the risk-free return be the same for debt and equity securities?

¹ Except for Hamada (1969, 1972), Sick (1990), and Taggart (1991) there appears to be little discussion in the literature of the risk-return implications of differential taxation of debt and equity. There is, on the other hand, an extensive analysis of equilibrium when dividends and capital gains are differentially taxed. See, for example, Long (1977), Elton and Gruber (1978), Litzenberger and Ramaswamy (1979), Singer (1979), Constantinides (1983), Constantinides (1984), Dammon and Green (1987), Ross (1987), Dammon (1988), Dammon, Dunn, and Spatt (1989), Bossaerts and Dammon (1994), Dammon and Spatt (1996), Basak and Gallmeyer (1999), and Green and Hollifield (1999). There is also a literature on international taxation (e.g., Black 1974 or Gordon and Varian 1989) which deals with market segmentation.

In part, the disagreement about the appropriate way to deal with differential taxation stems from the market segmentation associated with such taxation. When a tax code specifies differential taxation for asset classes it typically includes provisions to prevent tax arbitrage across these classes. Indeed, such restrictions may be necessary for the very existence of equilibrium.² Thus, differential taxation may imply different time values of money and different prices of risk across market segments. In the context of the CAPM, for example, this means that the SMLs of these market segments may have different intercepts and different slopes.

In this paper we build an equilibrium model which incorporates differential taxation both at the corporate and the personal levels. We prove necessary and sufficient conditions for the price of risk to be the same in both markets, even though the time value of money is typically different.

Using the same framework we address another related question: the interaction of leverage, taxation, and value in a differentially taxed world. The analysis of this issue is typically done using the adjusted present value (APV) approach. In the standard application of the APV, the tax effects of leverage are taken to be some fraction of the market value of debt. It is not clear, however, that the market value of the debt is a sufficient statistic for the impact of taxation on firm value. To see why this may be the case, consider two debt securities whose cash flows across states of nature are not the same but whose market values are equal. If the tax effect of debt financing is state dependent, it is not clear that the overall tax impact of these two debt issues will be the same, which is what the standard APV approach assumes. In the context of our preference-free model, we find necessary and sufficient conditions for the market value of debt to be a sufficient statistic from which the APV of a firm can be calculated. We show that these are the same conditions for the price of risk being the same in the debt and equity markets.

In our main analysis we take the supply of securities as given. To convert our results to practical use, we extend Miller's (1977) analysis, which allows corporations to adjust their supply of debt and

² Dammon and Green (1987) and Dammon (1988) show that, in some cases, restrictions on investor tax heterogeneity and on the span on the market make equilibrium possible without such restrictions.

equity based on the tax incentives they face, to an uncertain world. We show that when corporations can adjust the supply of debt and equity, the necessary and sufficient conditions for the equality of the price of risk and for the use of the standard APV approach are naturally satisfied. Under these conditions, we derive the relation between the risk-free returns of equity and debt. We also derive the appropriate risk adjustments for debt and equity and show that both can be assessed relative to the equity market portfolio. Based on the derived expressions for the risk-adjusted expected returns of debt and equity, we are able to correctly unlever betas, taking into account differential taxation, and to calculate the cost of capital with which corporate investments should be valued.

The structure of this paper is as follows. In Section 1 we describe our framework for analyzing equilibrium with differential asset taxation. In Section 2, we show, in this framework, that there are two pre-personal-tax SMLs, one for debt securities and one for equity securities. In this section we derive the conditions under which the price of risk in both markets is the same and show that the same conditions imply that the tax effect of leverage is a function of the value of a firm's debt only. In Section 3 we extend the Miller (1977) model and derive asset prices and risk-return relations in this setting. In Section 4 we apply the results to the calculation of asset betas and the weighted average cost of capital. Section 5 concludes.

1. The setting

We derive our results within a standard two-date state-preference model with multiple consumers (we also use the term “investors”) and multiple firms.³ We label the two dates 0 and 1 . At date 0 firms issue debt and equity securities and consumers invest their wealth in a portfolio of securities that pay off at date 1 . Uncertainty at date 1 is modeled by a finite number of states of nature $s \in \hat{I} \{1, \dots, S\}$.

³ The asset structure of the model is similar to that studied by DeAngelo and Masulis (1980), though our focus is, of course, different: DeAngelo and Masulis (1980) focus on the endogeneity of firm's effective tax rate under universal risk neutrality while we focus on the risk-return tradeoff.

In general, firms' choice of which securities to issue may affect the investment opportunities investors face. The role of capital structure in determining the span of the market has been extensively analyzed.⁴ In this paper we abstract from the question of which states of nature are spanned by traded securities and which are not by assuming that both the debt market and the equity market are complete—there exist S debt securities whose payoff spans the state space and there also exist S equity securities that span the state space. We term this situation *double completeness*. We abstract, for similar reasons, from other issues pertinent to the capital structure choice by assuming that there are no bankruptcy costs, agency costs, etc. Thus, the exclusive focus of our analysis is on the risk-return tradeoff in the presence of differential taxation.

Securities can be either *debt* or *equity*. The type of the security determines its taxation both at the investor level and at the corporate level. The payoff to the holder of a debt security is taxed when received by investor i in state s at the rate $\mathbf{t}_{PD}^i(s)$.⁵ Similarly, the payoff of an equity security is taxed when received by investor i in state s at the rate $\mathbf{t}_{PE}^i(s)$. We assume that, state by state, personal tax rates on debt income are higher than the personal tax rates on equity income: $1 > \mathbf{t}_{PD}^i(s) > \mathbf{t}_{PE}^i(s) \geq 0 \quad \forall i, s$.

The assumed double completeness of financial markets and the differential taxation of debt and equity income means that investors might wish to tax arbitrage across the debt and equity markets. If tax arbitrage is unrestricted, investor opportunity sets may be unbounded and equilibrium may fail to exist. Actual tax codes, however, prohibit unbounded tax arbitrage by restricting the tax deductibility of asset returns, which effectively restricts investors' asset holdings. To capture such restrictions in our setting, we assume that, state by state, investors may offset income received from securities with money paid on securities *of the same type* held short in computing their tax bill. Investors cannot, however, offset

⁴ See, for example, Allen and Gale (1988, 1991). Auerbach and King (1983) explicitly examine the interaction between market diversification and taxation.

⁵ We assume that the whole payoff is taxed. This is the appropriate equivalent in a one-period framework to taxation of rates of returns only in an infinite-horizon framework.

income received from one security type by payments on shorted securities of the another type.⁶ Note that despite the restrictions on asset holdings, since both the debt market and the equity market are complete, the investment opportunity sets of investors are not restricted by the deductibility constraints.

To focus on the risk-return tradeoff, we take the investment plans of firms as given. Formally, firms in our model are endowed with state-dependent, before-interest, before-corporate-tax income. This income is assumed to reflect each firm's optimal investment decisions. Firms' income is taxed at the rate t_C when income is positive. Since we are dealing with a single-period model, there is no carry backward or forward of losses; when corporate income is negative the corporation's tax rate is zero. A firm's payments to holders of debt securities are tax-deductible expenses while payments to holders of equity securities are not.

2. Equilibrium asset prices and risk-return tradeoff

Since investors are not restricted in deducting asset payoffs *within* a security class (while being restricted in *cross*-class deductability), equilibrium means the lack of arbitrage opportunities *within* an asset class.⁷ This means that, for all investors, there is no vector of portfolio holdings that allows after-tax arbitrage profits *within* a security class—either equity or debt. The implication of this are formally derived next.

Consider, first, equity securities. Let \tilde{Q}_E^j be the $1 \times S$ vector of state-dependent *before-personal tax* payoff of equity security j : $\tilde{Q}_E^j = \{Q_E^j(1), \dots, Q_E^j(S)\}$ and let \tilde{Q}_E be the $S \times S$ matrix of payoffs of S equity securities that span the state space (i.e., the rank of \tilde{Q}_E is S). To convert the before-personal-tax payoff matrix to an after-personal-tax matrix for investor i , define an $S \times S$ diagonal matrix, T_E^i in which

⁶ This assumption can be weakened somewhat, see Dammon and Green (1987) and Talmor (1989).

⁷ Even though tax deductability across asset classes is not permitted, lack of arbitrage also restricts the relative pricing of equity and debt securities. For our purposes we need to use only the restrictions on the relative pricing *within* security classes.

the s diagonal element is investor i 's after personal tax per dollar equity payoff, $[1 - t_{PE}^i(s)]$. The product $\tilde{Q}_E \cdot T_E^i$ is the net-of-personal-tax payoff matrix of equity securities that investor i sees given her personal tax rates.

Lack of arbitrage opportunities within the equity class of securities means that the prices of equity securities are such that any portfolio of equity securities with non-negative payoff at time I must have a non-negative cost. Let x be any S -dimensional column vector of equity security holdings. Lack of arbitrage means that $x^T \cdot \tilde{Q}_E \cdot T_E^i > 0$ implies $x^T \cdot V_E > 0$ where V_E is the S -dimensional column vector of prices of the S equity securities. By Farkas Lemma, lack of arbitrage opportunities means that there exists an S -dimensional column vector of equity state prices, P_E^i , such that $\tilde{Q}_E \cdot T_E^i \cdot P_E^i = V_E$. P_E^i is investor i 's vector of state prices used to price investor i 's *after-personal-tax* payoffs. Because investors differ in their personal tax rates, their state prices, P_E^i , differ as well. However, the *tax-adjusted* state prices, $T_E^i \cdot P_E^i$, which investors use to price the *before-personal-tax* payoffs of securities, are the same for all investors: Since \tilde{Q}_E is full rank (i.e., the equity market is complete), $T_E^i \cdot P_E^i = \tilde{Q}_E^{-1} \cdot V_E$ for *all* investors. Thus, market completeness and lack of arbitrage imply that all investors, regardless of their personal tax rates, price the market payoffs (i.e., pre-personal-tax payoffs) of all equity securities the same. We denote by $P_E = T_E^i \cdot P_E^i = \tilde{Q}_E^{-1} \cdot V_E$ the unique $S \times 1$ vector of equity securities' state prices for pre-personal-tax payoffs.

A similar analysis of the bond market means that lack of arbitrage entails, by Farkas Lemma, that there exist a unique $S \times 1$ vector of equity securities' state prices for pre-personal-tax payoffs. These state prices are given by $P_D = T_D^i \cdot P_D^i = \tilde{Q}_D^{-1} \cdot V_D$, where the different vectors are defined analogously to their definition for the equity market. Specifically, T_D^i is investor i 's state-dependent diagonal matrix of after-tax payoff, S_D^i is investor i 's state-price vector for pre-personal-tax payoffs, \tilde{Q}_D is the pre-personal-tax payoff matrix of the debt securities (which is full rank as this market segment is complete as well), and

V_D is the vector of debt securities' prices.

The objects of analysis in this paper are the equity and debt state prices, P_E and P_D , which investors use to uniquely price equity and debt pre-personal-tax payoffs, respectively. To simplify the exposition, we focus on equity and debt state securities. This entails no loss of generality: Because both the equity and the debt markets are, by assumption, complete, investors can create state securities by appropriately holding long and short positions in *only one security type*. This means that the payoffs of the securities held long and short to create state securities can be offset in the calculation of the taxable income in each state. Therefore, investor i may effectively buy a state- k debt security that pays \$1 *before personal tax* if and only if state k occurs. The cost of this security is $P_D(k)$, which is the same for all investors, and its payoff net of investor i 's personal taxation is $\$[1 - \mathbf{t}_{PD}^i(k)]$. Similarly, investor i can effectively buy an equity state- k security that costs $P_E(k)$ and pays $\$[1 - \mathbf{t}_{PE}^i(k)]$ net of investor i 's personal tax if and only if state k occurs.

The state prices derived in the previous paragraphs can also be derived in the framework of an equilibrium model in which investors maximize utility over a set of debt and equity state securities and in which no short-selling is allowed. The analysis of this investor optimization allows us to associate state prices with investor tax rates, which is useful in intuitively interpreting our results. Faced with complete equity and debt markets and the respective state-security prices, investor i chooses her holdings $x_E^i(s)$ and $x_D^i(s)$ in state- s equity securities and state- s debt securities, respectively, to solve the following utility maximization problem:

$$\begin{aligned}
 (1) \quad & \underset{x_E^i(s), x_D^i(s)}{\text{Max}} \quad u_i(c_{i0}) + \mathbf{d}_i E[u_i(\tilde{c})] = u_i(c_{i0}) + \mathbf{d}_i \sum_s \mathbf{p}(s) u_i(c_{is}) \\
 & = u_i(c_{i0}) + \mathbf{d}_i \sum_s \mathbf{p}(s) u_i \left[x_E^i(s) (1 - \mathbf{t}_{PE}^i(s)) + x_D^i(s) (1 - \mathbf{t}_{PD}^i(s)) \right] \\
 & \text{s.t.} \\
 & x_E^i(s), x_D^i(0) \geq 0, \quad c_{i0} + \sum_s x_E^i(s) \cdot P_E(s) + x_D^i(s) \cdot P_D(s) = W_i
 \end{aligned}$$

In (1) we let c_{i0} be consumer i 's consumption at date 0, $\tilde{c} = \{c_{i1}, \dots, c_{iS}\}$ be the consumer's after-tax consumption in each state at date 1, \mathbf{d}_i be her pure rate of time preference, and W_i be her wealth. The state probabilities, $\mathbf{p}(s)$, need not be homogeneous.

An individual i will be indifferent between purchasing a state- s *equity* security and purchasing a state- s *debt* security if and only if a given expenditure at date 0 on state- s securities gives the same state- s consumption whether done by purchasing equity securities or debt securities:

$$(2) \quad x_E P_E(s) = x_D P_D(s) \quad \Leftrightarrow \quad (1 - \mathbf{t}_{PD}^i(s))x_D = (1 - \mathbf{t}_{PE}^i(s))x_E.$$

Manipulating this condition shows that consumer i is indifferent between state- s equity and debt if and only if:

$$(3) \quad \frac{(1 - \mathbf{t}_{PD}^i(s))}{P_D(s)} = \frac{(1 - \mathbf{t}_{PE}^i(s))}{P_E(s)}$$

In other words investor indifference between buying state- s debt or state- s equity obtains if and only if cost equivalence implies consumption equivalence. Consumer i will prefer a state- s debt security instead

of a state- s equity security when $\frac{(1 - \mathbf{t}_{PD}^i(s))}{P_D(s)} > \frac{(1 - \mathbf{t}_{PE}^i(s))}{P_E(s)}$; she will prefer a state- s equity security

instead of a state- s debt security when $\frac{(1 - \mathbf{t}_{PD}^i(s))}{P_D(s)} < \frac{(1 - \mathbf{t}_{PE}^i(s))}{P_E(s)}$.

Let $\mathbf{j}(s) = \frac{P_D(s)}{P_E(s)}$ denote the ratio of debt and equity state prices. We assume that for each

state s there exists a consumer $i(s)$ who is indifferent between purchasing state- s debt and equity securities. A sufficiently rich range of relative tax rates will imply satisfaction of this assumption, which we refer to as the ‘‘continuity of tax rates’’ assumption. From the optimization problem (1) it follows that:

$$(4) \quad \mathbf{j}(s) = \frac{(1 - \mathbf{t}_{PD}^{i(s)}(s))}{(1 - \mathbf{t}_{PE}^{i(s)}(s))}.$$

Given expression (4), we identify each state's relative pricing of equity and debt income with investor $i(s)$'s ratio of net-of-personal-tax state- s income.

Since investors are, by assumption, taxed less heavily on debt income than on equity income, $0 < \mathbf{j}(s) < 1$ and hence $P_D(s) = \mathbf{j}(s)P_E(s) < P_E(s)$. In words: Since debt income is taxed more heavily than equity income, the price of a debt security that pays one pre-personal-tax dollar if state s occurs is lower than the price of an equity state security for the same state.

We now turn to the examination of the risk-return relations of the equity and the debt markets. Consider an arbitrary equity security that pays a random before-personal-tax payoff of \tilde{Q}_E , where \tilde{Q}_E is a vector of state-dependent payoffs $\tilde{Q}_E = \{Q_E(s)\}$. Since an equity security that pays off \tilde{Q}_E at time 1 is effectively a bundle of equity state securities, its price at time 0 is $V_E = \sum_s Q_E(s)P_E(s)$. Similarly, the price of a debt security that pays $\tilde{Q}_D = \{Q_D(s)\}$ at time 1 is given by $V_D = \sum_s Q_D(s)P_D(s) = \sum_s Q_D(s)\mathbf{j}(s)P_E(s)$.

A security of particular importance is the risk-free security—a security that pays a state-independent payoff. This security determines the risk-free rate of return. When debt and equity are differentially taxed, it is necessary to distinguish between the risk-free return of *debt* securities and the risk-free return of *equity* securities. Using the price of a security that pays off a dollar in every state, we get that the risk-free *equity rate of interest* is given by $Rf_E \equiv 1 + rf_E = 1 / \sum_s P_E(s)$, and the risk-free *debt rate of interest* is given by $Rf_D \equiv 1 + rf_D = 1 / \sum_s P_D(s)$.

The risk-free returns for equity and debt reflect the relative tax advantage of equity income over debt income: Since equity income is less heavily taxed than debt income, the before-tax risk-free rate of return for equity securities is lower than the before-tax risk-free rate of return for debt securities. Formally, since $P_D(s) < P_E(s) \quad \forall s$:

$$(5) \quad Rf_D = \frac{1}{\sum_s P_D(s)} = \frac{1}{\sum_s j(s) P_E(s)} > \frac{1}{\sum_s P_E(s)} = Rf_E$$

Next we consider the expected returns of *risky* securities. We discuss distributions of the *total* returns— \tilde{R} —rather than distributions of the *rates* of return— $\tilde{r} = \tilde{R} - 1$. We do this because the expressions derived in a single period model for *total* returns and their relations to values have the same form as the expressions and relations obtained in an infinite horizon models using rates of return.

Proposition 1: There are two security market lines (SMLs) in this model, one for debt securities and one for equity securities.

Proof: Using a technique first introduced by Beja (1972), we rewrite the prices of a risky equity security that has a before-personal-tax payoff vector \tilde{Q}_E as:

$$(6) \quad \begin{aligned} V_E &= \sum_s P_E(s) Q_E(s) = \sum_s \mathbf{p}(s) \cdot \frac{P_E(s)}{\mathbf{p}(s)} \cdot Q_E(s) \\ &= E \left[\left(\frac{P_E(s)}{\mathbf{p}(s)} \right) \cdot Q_E(s) \right] = E \left(\frac{P_E(s)}{\mathbf{p}(s)} \right) \cdot E(\tilde{Q}_E) + Cov \left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{Q}_E \right) \\ &= \frac{1}{Rf_E} E(\tilde{Q}_E) + Cov \left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{Q}_E \right) \end{aligned}$$

Since $\tilde{R}_E = \tilde{Q}_E / V_E$, simple manipulation of the above expression gives:

$$(7) \quad \begin{aligned} Rf_E &= \frac{E(\tilde{Q}_E)}{V_E} + \frac{Cov \left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{Q}_E \right) Rf_E}{V_E} = E(\tilde{R}_E) + Cov \left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{R}_E \right) Rf_E \\ \Leftrightarrow \quad E(\tilde{R}_E) &= Rf_E - Cov \left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{R}_E \right) Rf_E \end{aligned}$$

Bringing Rf_E inside the parentheses in (7) to get the *equity pricing kernel*, $\tilde{M}_E \equiv \frac{Rf_E \cdot \tilde{P}_E}{\tilde{\mathbf{p}}}$, gives

the equity market SML:

$$(8) \quad E(\tilde{R}_E) = Rf_E - Cov\left(\frac{Rf_E \cdot \tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{R}_E\right) \equiv Rf_E - Cov(\tilde{M}_E, \tilde{R}_E)$$

Performing a similar exercise for a risky debt security gives the debt market SML:

$$(9) \quad E(\tilde{R}_D) = Rf_D - Cov\left(\frac{\tilde{P}_D}{\tilde{\mathbf{p}}}, \tilde{R}_D\right) Rf_D \equiv Rf_D - Cov(\tilde{M}_D, \tilde{R}_D),$$

where \tilde{M}_D is the debt pricing kernel.

Since $Rf_E < Rf_D$, the equity-market SML differs from the debt-market SML. Generically, the price of risk in these markets will also be different. ||

Equations (8) and (9) for the expected returns of risky debt and equity securities can be viewed as the debt and equity security market lines (SMLs) with the pricing kernels $\tilde{M}_D \equiv (Rf_D \cdot \tilde{P}_D)/\tilde{\mathbf{p}}$ and $\tilde{M}_E \equiv (Rf_E \cdot \tilde{P}_E)/\tilde{\mathbf{p}}$, respectively, replacing the returns on the respective market portfolios in the standard CAPM. Note that the pricing kernels are used to determine the risk premiums of the debt and equity securities as they are added to the risk-free benchmarks (in equations 8 and 9). In Section 4 we rederive these SMLs in their more familiar form by using traded assets that are perfectly correlated with the pricing kernels.

The two equations (8) and (9) imply that in general there are *two separate SMLs*, one for debt securities and one for equity securities and that these two SMLs have different intercepts and different slopes. This means that the expected rates of return of an equity security and a debt security with identically distributed before-personal-tax payoffs are, in general, different. The different pricing of debt and equity securities is due to the differential taxation of debt and equity and is sustained by the restrictions on tax arbitrage, restrictions which entail a segmentation between investors who hold state-*s* debt and equity securities.

Proposition 2: The price of risk is the same for debt and equity securities (i.e., the equity SML and the debt SML are parallel with different intercepts) if and only if for every state s $\mathbf{j}(s) = \mathbf{j}$.

Proof:

Sufficiency: Assume that $\mathbf{j}(s) = \mathbf{j}$ for every state s . Then

$$\begin{aligned}
 (10) \quad E(\tilde{R}_D) &= Rf_D - Cov\left(\frac{\tilde{P}_D}{\tilde{\mathbf{p}}}, \tilde{R}_D\right) Rf_D = Rf_D - Cov\left(\frac{\mathbf{j} \tilde{P}_D}{\tilde{\mathbf{p}}}, \tilde{R}_D\right) \frac{Rf_D}{\mathbf{j}} \\
 &= Rf_D - Cov\left(\frac{\tilde{P}_E}{\tilde{\mathbf{p}}}, \tilde{R}_D\right) Rf_E = Rf_D - Cov\left(\frac{\tilde{P}_E Rf_E}{\tilde{\mathbf{p}}}, \tilde{R}_D\right) = Rf_D - Cov(\tilde{M}_E, \tilde{R}_D)
 \end{aligned}$$

Necessity: Assume that the price of risk is the same (i.e., that the SMLs are parallel with possibly different intercepts). This means that the risk premium of any debt security can be determined by using either the debt or the equity kernel:

$$(11) \quad E(\tilde{R}_D) - Rf_D = Cov(\tilde{M}_E, \tilde{R}_D) = Cov(\tilde{M}_D, \tilde{R}_E).$$

In particular, this is true for a pure state- s debt security. Let $1_D(s)$ be a debt security that pays off one dollar in state s only. For this security the covariance of its payoff with the *equity* market pricing kernel is:

$$(12) \quad Cov(\tilde{M}_E, 1_D(s)) = Cov\left(\frac{P_E(s)Rf_E}{\mathbf{p}(s)}, 1_D(s)\right) = P_E(s)Rf_E - \mathbf{p}(s).$$

Similarly, the covariance of this security's payoff with the *debt* market pricing kernel is:

$$(13) \quad Cov(\tilde{M}_D, 1_D(s)) = P_D(s)Rf_D - \mathbf{p}(s).$$

Equating the two above expressions yields:

$$(14) \quad \frac{P_D(s)}{P_E(s)} = \frac{Rf_E}{Rf_D} \text{ for all states } s.$$

Since $\frac{P_D(s)}{P_E(s)} = \mathbf{j}(s)$, $\mathbf{j}(s) = \frac{Rf_E}{Rf_D}$ for all s if the price of risk is the same under both kernels. ||

Later, we use the following corollary to Proposition 2:

Corollary: The pricing kernel for debt and equity securities is the same if and only if $\mathbf{j}(s) = \mathbf{j}$ for all s .

Proof: The equity and debt pricing kernels are $M_E(s) = \frac{P_E(s)Rf_E}{\mathbf{p}(s)}$, $M_D(s) = \frac{P_D(s)Rf_D}{\mathbf{p}(s)}$. Since

under the conditions of the corollary $P_D(s) = \mathbf{j}P_E(s)$, the result follows immediately from the definitions of the risk-free rates Rf_E and Rf_D . \parallel

Proposition 2 shows that the necessary and sufficient condition for the SMLs of debt and equity securities to be parallel is that *state by state* the relative pricing of debt income and equity income is the same. Thus, Proposition 2 characterizes the conditions for the price of risk to be the same for any before-personal-tax return distribution. The proposition states that a necessary and sufficient condition for the price of risk to be the same for equity and debt is that the marginal, relative, tax rate, $\mathbf{j}(s)$, is state independent. While the marginal investor in each state, $i(s)$, need not be the same individual, the marginal tax rates on equity and debt incomes should have a constant ratio in *all* states. In the next section we show that this happens in an extension of the Miller (1977) equilibrium.

Proposition 2 also provides a necessary and sufficient condition to answer one of the questions posed in the introduction: If the breakeven ratio of tax rates is state-independent, i.e. $\mathbf{j}(s) = \mathbf{j}$ for all s , then the price of risk is independent of the classification of a security as debt or equity. The intuition is that when the relative pricing of debt and equity incomes is state independent, the relative pricing of debt and equity securities reflects the same state-independent relative taxation. Hence, when computing the covariance of an arbitrary return with either the pricing kernel of debt or equity, the covariance term captures the relative pricing of debt and equity state securities.

Proposition 2 establishes the necessary and sufficient conditions for the price of risk in the debt and equity markets to be the same. As we show in Proposition 3 below, the condition of Proposition 2—

i.e., that the break-even tax rate $\mathbf{j}(s)$ is state independent—is also related to the standard adjusted present value (APV) approach for the valuation of a levered versus an unlevered firm. Suppose we write—as in the APV approach to valuing leverage— $\Delta V \equiv V_L - V_U = K \cdot V_D$, where V_L and V_U are, respectively, the values of the same firm with and without leverage, V_D is the market value of the levered firm's debt, and K is some constant that reflects the tax-related valuation impact of leverage. Implicit in these formulas is the assumption that the market value of the debt, V_D , is a sufficient statistic for the valuation impact of leverage; otherwise K should depend on the particular distribution of the debt payments.

Recall that \mathbf{t}_C denotes the statutory corporate tax rate. Then:

Proposition 3: The adjustment of the value of the firm for the tax effect of leverage is independent of the shape of the distribution of the debt payments (i.e., the market value of the debt is a sufficient statistic to compute the tax effect of leverage) if and only if $\frac{P_E(s) \cdot (1 - \mathbf{t}_C)}{P_D(s)}$ is state-independent. Since

$$\frac{P_E(s)}{P_D(s)} = \frac{1}{\mathbf{j}(s)},$$

this condition is equivalent to the state-independence of $\mathbf{j}(s)$.

Proof: Let the firm's after-corporate-tax operating cash flows in state s be denoted by $FCF(s)$ and the state- s payments to the debt holders be denoted by $CF_D(s)$. Then, the value of the unlevered firm is

$$V_U = \sum_s P_E(s) \cdot FCF(s)$$

and the value of the levered firm is

$$V_L = V_E + V_D = \sum_s P_E(s) \cdot \{FCF(s) - CF_D(s) \cdot (1 - \mathbf{t}_C)\} + \sum_s P_D(s) \cdot CF_D(s).$$

A simple manipulation gives:

$$\begin{aligned} \Delta V \equiv V^L - V^U &= \sum_s CF_D \cdot \{P_D(s) - P_E(s) \cdot (1 - \mathbf{t}_C)\} \\ (15) \quad &= \sum_s CF_D \cdot P_D(s) \cdot \left\{1 - \frac{P_E(s) \cdot (1 - \mathbf{t}_C)}{P_D(s)}\right\} \end{aligned}$$

Sufficiency: If $1 - \frac{P_E(s) \cdot (1 - t_C)}{P_D(s)} = K \forall s$, then equation (15) simplifies to:

$$\Delta V = \sum_s CF_D \cdot P_D(s) \cdot \left\{ 1 - \frac{P_E(s) \cdot (1 - t_C)}{P_D(s)} \right\} = \sum_s CF_D \cdot P_D(s) \cdot K = K \cdot V_D .$$

Necessity: If $\Delta V = K \cdot V_D$ for all debt payoff patterns, then it is also true for any specific debt state security. Recall that $1_D(s)$ denotes the payoff of a state- s debt security (i.e., a debt security that pays off one dollar in state s only). The impact of this security on the value of the firm is:

$$\Delta V = \sum_s 1_D(s) \cdot P_D(s) \cdot \left\{ 1 - \frac{P_E(s) \cdot (1 - t_C)}{P_D(s)} \right\} = P_D(s) - P_E(s) \cdot (1 - t_C)$$

By assumption

$$\Delta V = K \cdot V_D$$

where K is the distribution-independent tax effect of leverage. Since the value of a state- s debt security is $P_D(s)$, we get:

$$\begin{aligned} \Delta V &= P_D(s) - P_E(s) \cdot (1 - t_C) = K \cdot P_D(s) \\ \Rightarrow \quad \left\{ 1 - \frac{P_E(s) \cdot (1 - t_C)}{P_D(s)} \right\} &= K \quad \forall s . \quad \parallel \end{aligned}$$

Proposition 3 establishes necessary and sufficient conditions under which one can apply the standard APV approach of accounting for the tax-induced valuation impact of leverage. It shows that the standard value additivity approach to valuing leverage holds if and only if the relative personal taxation of debt and equity income $\mathbf{j}(s)$ is state-independent. By Proposition 2 this condition is also necessary and sufficient for a single price of risk for debt and equity returns (i.e., for parallel debt and equity SMLs). Proposition 3, therefore, means that the two standard tools of corporate finance in the treatment of differential asset taxation:

- Adjusting the risk-free returns of assets while keeping the price of risk equal across security classes, and
- Taking the impact of differential taxation to be independent of the payoff pattern of the firm's debt in Adjusted Present Values

share a common underlying implicit assumption—that the relative taxation of debt and equity income at the personal level are state-independent. While this condition may seem unreasonably restrictive, in the next section we show that, extending the intuition of Miller (1977), this condition is rather a natural outcome when corporations can adjust their security offerings in response to tax incentives.

3. Asset prices in an extended version of the Miller equilibrium

In general, the marginal relative tax rates of debt and equity incomes need not be the same in all states. Consequently, Propositions 2 and 3 imply that the standard tools of corporate finance, unlevering betas assuming equal price of risk across security types and APV, cannot be applied. Yet, because corporations can adjust the amounts of debt and equity securities they issue as long as they find it advantageous to do so, in equilibrium the relative taxation of debt and equity incomes can become state-independent. Indeed, we now show that extending the Miller (1977) analysis to an uncertain world re-establishes the viability of these tools. The difference between Miller's setting and ours is that we explicitly account for risk and allow firms to issue state-dependent securities. Hence, we extend the Miller argument to a state-by-state basis.

In this section, we allow corporations to adjust the menu of securities they issue. This means that it is no longer guaranteed that double completeness of both the equity and the debt markets obtains. Thus, we assume that double completeness does obtain and examine the effect of corporate adjustment of the mix of securities they offer on returns and prices. It is worth noting that the resulting equilibrium conditions are such that double spanning is a likely. This is because, in the extended Miller equilibrium we analyze, firms are indifferent between issuing debt or equity securities, which makes it unlikely that

any security that has potential demand will not be offered. Moreover, since corporations may issue any bundle of debt and equity securities, they can be viewed as effectively being able to offer state securities, debt or equity. Thus, to simplify exposition, we consider corporate issuance of state securities (and not the bundles that create them).

Recall that corporate income in state s is taxed at the rate t_C if the corporation's income is positive and zero otherwise. We let I_p denote the indicator function that takes the value 1 if the firm is tax-wise profitable and 0 otherwise. Consider $\$I$ of before-corporate-tax income in state s at time t . If paid out to equity holders, this $\$I$ will be taxed at the corporate level and will have a value of $P_E(s) \cdot (1 - t_C \cdot I_p)$. If paid out as debt, this $\$I$ will not be taxed at the corporate level and will have a value of $P_D(s) \cdot I = j(s) \cdot P_E(s) \cdot I$.

The following result holds:

Proposition 4: Corporations never issue more state-contingent debt than their before-corporate-tax and before-debt-payment income in the state. Every corporation in every state in which it is profitable issues (up to its state-contingent income):

debt if $j(s) > 1 - t_C$
equity if $j(s) < 1 - t_C$,
either debt or equity if $j(s) = 1 - t_C$.

Proof:

Since effective issuance of state securities is possible (possibly requiring the issuance of a bundle of securities), corporate value is maximized by issuing the security that maximizes the market value of the corporation's before-corporate-tax income *in each state*. This is achieved by the following state-by-state comparison:

$$P_E(s) \cdot [1 - t_C \cdot I_p] \stackrel{<}{=} P_D(s) = j(s) \cdot P_E(s)$$

In states where the corporation has exhausted its pre-debt-payment income (i.e., when the corporation issues more debt than its state-contingent income), $I_p = 0$ and equity issuance maximizes the value of this state's remaining before-corporate-tax income. When the corporation is profitable, $I_p = 1$ and the result trivially follows. ||

Note that the relative price of state- s debt and equity income corporations face, $\mathbf{j}(s)$, is *the same for all corporations*. Proposition 4, therefore, means that in every state all corporations that are profitable choose the same financing pattern unless the choice of finance is a matter of indifference. In particular, so long as $\mathbf{j}(s) > 1 - \mathbf{t}_C$ corporations continue to issue state- s debt. We assume, as in Miller (1977), that there are individuals with relative tax rates (i.e., $[1 - \mathbf{t}_{PD}^i(s)]/[1 - \mathbf{t}_{PE}^i(s)]$) both above and below $(1 - \mathbf{t}_C)$ in every state. The state- s debt the corporations offer is first sold to high $[1 - \mathbf{t}_{PD}^i(s)]/[1 - \mathbf{t}_{PE}^i(s)]$ investors. To attract additional, low $[1 - \mathbf{t}_{PD}^i(s)]/[1 - \mathbf{t}_{PE}^i(s)]$ investors to buy state- s debt, the state- s price of debt has to decline relative to the state- s price of equity (i.e., $\mathbf{j}^i(s)$ has to decline). If in state s the aggregate corporate before-tax income is not exhausted by issuing debt only to investors with $[1 - \mathbf{t}_{PD}^i(s)]/[1 - \mathbf{t}_{PE}^i(s)] > (1 - \mathbf{t}_C)$, by Proposition 4 state- s debt will be issued until $\mathbf{j}(s) = (1 - \mathbf{t}_C)$ (for at this $\mathbf{j}(s)$ corporations have no incentive to issue additional debt).

Assumption: capital structure endogeneity

We assume that in equilibrium, in each state s there exists at least one firm with endogenous capital structure.

The capital structure endogeneity assumption means that *in each state* there is at least one firm that is indifferent between issuing a dollar of state- s debt or a dollar of state- s equity. Note that the firm with internal capital structure need not be the same in all states. But in every state there is at least one

profitable firm that has not exhausted its debt capacity. Given Proposition 4, this assumption is a sufficient condition for a state-by-state equilibrium that is equivalent to Miller's (1977) equilibrium. We call this an Extended Miller Equilibrium (EME):

Corollary: Under the assumption of capital structure endogeneity (i.e., in an EME), $\mathbf{j}(s) = 1 - \mathbf{t}_C$ in all states.

The corollary equates the relative pricing of state- s debt and equity securities with the statutory corporate tax rate. In particular, it implies that the relative pricing of state- s debt and equity securities is state-independent: $\mathbf{j}(s) = 1 - \mathbf{t}_C = \mathbf{j}$ for all s . Given the equality of the relative pricing of debt and equity in all states, the risk-free rates are related in the same manner:

$$(16) \quad Rf_D = \frac{1}{\sum_s P_D(s)} = \frac{1}{\sum_s \mathbf{j}(s) P_E(s)} = \frac{1}{\mathbf{j} \sum_s P_E(s)} = \frac{1}{(1 - \mathbf{t}_C) \sum_s P_E(s)} = \frac{1}{(1 - \mathbf{t}_C)} Rf_E.$$

Thus, in an EME we can answer another question posed in the introduction: What are the relative risk-free returns of equity and debt securities? In an EME, the risk-free benchmark return for *equity* securities is the risk-free return for *debt* securities *after corporate tax*. Moreover, note that in an EME the conditions of Propositions 2 and 3 are met. Therefore, in an EME, the price of risk is the same in the debt and the equity markets. In other words, in an EME, the slopes of the equity and the debt SMLs are equal. In Proposition 5 we fully specify the SMLs of debt and equity securities in an EME. Before we do this, however, we first convert the state-preference setting into an equivalent standard CAPM setting where the pricing kernel is identified with a traded asset and the risk-return tradeoff for debt and equity securities is estimable.

Consider an equity asset, actual or a portfolio of other equity securities, whose return is perfectly correlated with the equity pricing kernel: $\tilde{M}_E = a + b\tilde{R}_{mE}$ where \tilde{R}_{mE} is the return on this portfolio.

We call this portfolio “the equity market portfolio.”⁸ Using the equity market portfolio we can derive a standard CAPM:

$$(17) \quad \begin{aligned} E(\tilde{R}_E) &= Rf_E - Cov(\tilde{M}_E, \tilde{R}_E) = Rf_E - Cov(a + b\tilde{R}_m, \tilde{R}_E) \\ &= Rf_E - bCov(\tilde{R}_m, \tilde{R}_E) \end{aligned}$$

Since, in particular, this relation is true for the equity market portfolio itself, it follows that

$$(18) \quad E(\tilde{R}_{mE}) = Rf_E - bCov(\tilde{R}_{mE}, \tilde{R}_{mE}) \quad \Rightarrow \quad -b = \frac{E(\tilde{R}_{mE}) - Rf_E}{\mathbf{s}_{mE}^2},$$

where \mathbf{s}_{mE}^2 is the variance of the return of the equity market portfolio. Substituting into the expression for the pricing of equity securities gives the standard equity CAPM:

$$(19) \quad \begin{aligned} E(\tilde{R}_E) &= Rf_E + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_E)}{\mathbf{s}_{mE}^2} [E(\tilde{R}_{mE}) - Rf_E] \\ &\equiv Rf_E + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_E)}{\mathbf{s}_{mE}^2} \cdot \Pi_{mE} \equiv Rf_E + \mathbf{b}_E \cdot \Pi_{mE} \end{aligned}$$

where Π_{mE} is the risk premium on the equity market portfolio and \mathbf{b}_E is the beta coefficient of the return of an *equity* security vis-a-vis the return of the *equity* market portfolio.

Similarly using a bond, actual or a portfolio of other bonds, whose returns are perfectly correlated with the debt-market pricing kernel, we get a similar relation for the debt market:

$$(20) \quad \begin{aligned} E(\tilde{R}_D) &= Rf_D - Cov(\tilde{M}_D, \tilde{R}_D) = Rf_D - Cov(a + b\tilde{R}_{mD}, \tilde{R}_D) \\ &= Rf_D - bCov(\tilde{R}_{mD}, \tilde{R}_D) = Rf_D + \frac{Cov(\tilde{R}_{mD}, \tilde{R}_D)}{\mathbf{s}_{mD}^2} [E(\tilde{R}_{mD}) - Rf_D] \\ &\equiv Rf_D + \frac{Cov(\tilde{R}_{mD}, \tilde{R}_D)}{\mathbf{s}_{mD}^2} \cdot \Pi_{mD} \equiv Rf_D + \mathbf{b}_D \cdot \Pi_{mD} \end{aligned}$$

where Π_{mD} is the risk premium on the debt market portfolio and \mathbf{b}_D is the beta coefficient of the return of a *debt* security vis-a-vis the *debt* market portfolio.

⁸ Obviously, nothing in our assumptions guaranties that this is the market-weighted portfolio of investor equity holdings.

Recall that the SMLs of the two markets will, in general, have different intercepts and different slopes. Using the results of Propositions 2, 4, and 5, we can relate the pricing parameters of the two SML equations:

Proposition 5: In an EME, the SMLs of equity and debt securities are:

The equity SML:

$$\begin{aligned}
 E(\tilde{R}_E) &= Rf_D \cdot (1 - t_C) + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_E)}{\mathbf{s}_{mE}^2} \left[E(\tilde{R}_{mE}) - Rf_D \cdot (1 - t_C) \right] \\
 (21) \quad &= Rf_D \cdot (1 - t_C) + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_E)}{\mathbf{s}_{mE}^2} \cdot \Pi_{mE} \\
 &= Rf_D \cdot (1 - t_C) + \mathbf{b}_E \cdot \Pi_{mE}
 \end{aligned}$$

The debt SML

$$\begin{aligned}
 E(\tilde{R}_D) &= Rf_D + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_D)}{\mathbf{s}_{mE}^2} \left[E(\tilde{R}_{mE}) - Rf_D \cdot (1 - t_C) \right] \\
 (22) \quad &= Rf_D + \frac{Cov(\tilde{R}_{mE}, \tilde{R}_D)}{\mathbf{s}_{mE}^2} \cdot \Pi_{mE} \\
 &= Rf_D + \mathbf{b}_D \cdot \Pi_{mE}
 \end{aligned}$$

Note that the risk premia are measured *in both markets* relative to the *equity* market portfolio, $\Pi_{mE} = E(\tilde{R}_{mE}) - Rf_E \cdot (1 - t_C)$, and that betas are estimated *for both security types* vis-a-vis the *equity* market index.

Proof: In an EME, $\tilde{R}_{mD} = \tilde{R}_{mE} / (1 - t_C)$. Thus, in equation (20), the term $\mathbf{b}_D \cdot \Pi_{mD}$ becomes

$$\begin{aligned}
 \frac{Cov(\tilde{R}_{mD}, \tilde{R}_D)}{\mathbf{s}_{mD}^2} \left[E(\tilde{R}_{mD}) - Rf_D \right] &= \frac{Cov(\tilde{R}_{mE} / (1 - t_C), \tilde{R}_D)}{\mathbf{s}_{mD}^2} \left[E(\tilde{R}_{mE}) / (1 - t_C) - Rf_E / (1 - t_C) \right] \\
 &= \frac{Cov(\tilde{R}_{mE}, \tilde{R}_D)}{\mathbf{s}_{mD}^2} \left[E(\tilde{R}_{mE}) - Rf_E \right]
 \end{aligned}$$

Since $Rf_E = Rf_D \cdot (1 - t_C)$, this established the result. ||

Proposition 5 characterizes the relation between the risk-free return of equity securities and the risk-free return of debt securities and the risk adjustments of both SMLs in an EME. The intercepts of the SMLs are as given above: Rf_E and $(1-t_C)Rf_D$, respectively. Since $\mathbf{j}(s)$ is state independent in an EME, by Proposition 2, the debt and equity SMLs are parallel. Proposition 5 shows that the common slope of the equity and debt SMLs is $E(\tilde{R}_{Me}) - Rf_D(1-t_C)$.⁹

Proposition 5 shows that the risk premium for both equity securities and debt securities can be estimated relative to the equity market portfolio and with the equity market portfolio's risk premium. This is despite the fact the the equity and the debt markets are segmented by differential personal taxation. The reason is that corporate adjustments of their security offerings bridges the personal taxation gap across market segments.

4. Calculating the weighted average cost of capital (WACC)

One of the basic tools of practical corporate finance is the unlevering of the rates of returns estimated for the individual securities the firm has issued to compute the weighed average cost of capital, the WACC. The WACC is used to estimate the firm's cost of capital with which projects are evaluated. An equivalent computation is to unlever the beta estimates of these securities and use an "asset" beta and an "asset" SML to estimate the cost of capital. Obviously, an "asset" SML exists only if the SMLs of the individual securities—debt and equity—are parallel; that is, only if the price of risk is the same for debt and equity securities does it make sense to average the risk of debt and equity into a unique asset risk. This is because when the risk premia of debt and equity are not the same, it is not clear what price of risk to apply to the "unlevered" beta estimate.

⁹ Hamada and Scholes (1985) suggest an after-tax equity SML which is similar to that of Propostion 6; Taggart (1991) also presents a similar equity SML. Neither of these papers has the capital markets equilibrium characterization which underlies our results, nor do these papers examine the case of risky debt. Sick (1990) obtains results similar to Proposition 5, although not in a general equilibrium framework.

In the preceding sections we derive the necessary and sufficient for the price of risk to be the same in the debt and equity markets. We also derive the SML equations when corporations may adjust the mix of debt and equity securities they issue, i.e., in an EME. The SML equations we obtain for debt and equity securities in an EME allow us to derive the correct way to unlever equity and debt betas and estimate the cost of capital using the unlevered beta and the asset SML:

Proposition 6: Consider a firm financed with a proportion $x_E \equiv \frac{V_E}{V_D + V_E}$ of equity and $x_D = 1 - x_E$

debt, where V_E and V_D denote the market values of the firm's equity and debt, respectively. In an EME, the estimated equity and debt betas are unlevered to an *asset* beta by:

$$\mathbf{b}_{Assets} = x_E \cdot \mathbf{b}_{Equity} + x_D \cdot \mathbf{b}_{Debt} \cdot (1 - \mathbf{t}_C).$$

Furthermore, the asset SML used in calculating the required return on the assets relative to the estimated asset betas is the *equity* SML:

$$E(\tilde{R}_{Assets}) = Rf_D \cdot (1 - \mathbf{t}_C) + \mathbf{b}_{Assets} \cdot \Pi_{mE}$$

Proof: The firm's assets generate a state- s cash flow of $CF_A(s)$ before debt payments and before corporate taxes. Denote the firm's *free cash flow* by $FCF(s) = CF_A(s) \cdot (1 - \mathbf{t}_C)$. The firm's debt cash flow is $CF_D(s)$ in state s . The value of equity and debt is given by:

$$(25) \quad \begin{aligned} V_D &= \sum_s P_D(s) CF_D(s) = \sum_s (1 - \mathbf{t}_C) P_E(s) CF_D(s) \\ V_E &= \sum_s P_E(s) CF_E(s) = \sum_s P_E(s) (FCF(s) - (1 - \mathbf{t}_C) CF_D(s)) \end{aligned}$$

A simple summation shows that in an EME the value of the firm is invariant to leverage:

$$V_F = V_D + V_E = \sum_s P_E(s) FCF(s)$$

This further implies that the value of the firm can be derived assuming the firm is all equity financed, i.e., from the firm's free cash flows valued by the *equity* state prices. Thus, the value of the firm is computed using the *equity* SML and an asset beta estimated relative to the *equity* market portfolio:

$$(26) \quad E(r_A) = Rf_E + \mathbf{b}^{Assets} \cdot \Pi_{ME} \quad \text{and} \quad \mathbf{b}^{Assets} = \frac{\text{Cov}\left[\frac{FCF}{(V_E + V_D)}, R_{ME}\right]}{\text{Var}(R_{ME})}.$$

Now consider the case where we estimate the asset \mathbf{b}^{Assets} from a weighted average of the beta of debt (\mathbf{b}^{Debt}) and the beta of equity (\mathbf{b}^{Equity}). Compute the betas of the debt and the equity:

$$(27) \quad \mathbf{b}^{Debt} = \frac{\text{Cov}\left(\frac{CF_D}{V_D}, \tilde{R}_{ME}\right)}{\text{Var}(\tilde{R}_{ME})} = \frac{1}{V_D} \frac{\text{Cov}(CF_D, \tilde{R}_{ME})}{\text{Var}(\tilde{R}_{ME})}$$

$$\mathbf{b}^{Equity} = \frac{\text{Cov}\left(\frac{CF_E}{V_E}, \tilde{R}_{ME}\right)}{\text{Var}(\tilde{R}_{ME})} = \frac{1}{V_E} \frac{\text{Cov}(FCF - CF_D(1 - t_C), \tilde{R}_{ME})}{\text{Var}(\tilde{R}_{ME})}$$

We want to compute the asset \mathbf{b}^{Assets} as a weighted average of the debt and the equity \mathbf{b} s, using market values in the weights. Letting K be some constant of proportionality on the debt \mathbf{b} :

$$(28) \quad \frac{V_E}{V_D + V_E} \cdot \mathbf{b}^{Equity} + \frac{V_D}{V_D + V_E} \cdot K \cdot \mathbf{b}^{Debt}$$

$$= \left(\frac{1}{V_D + V_E}\right) \cdot \left[\frac{\text{Cov}(FCF - CF_D(1 - t_C), \tilde{R}_{ME})}{\text{Var}(\tilde{R}_{ME})}\right] + K \cdot \left(\frac{1}{V_D + V_E}\right) \cdot \left[\frac{\text{Cov}(CF_D, \tilde{R}_{ME})}{\text{Var}(\tilde{R}_{ME})}\right]$$

This should equal $\mathbf{b}^{Assets} = \frac{\text{Cov}\left[\frac{FCF}{(V_E + V_D)}, R_{ME}\right]}{\text{Var}(R_{ME})}$, which is easily seen to be true for all beta

estimates if and only if $K = (1 - t_C)$. The appropriate averaging of the equity and debt betas to an asset beta is:

$$\mathbf{b}^{Assets} = \left(\frac{V_E}{V_D + V_E}\right) \cdot \mathbf{b}^{Equity} + \left(\frac{V_D}{V_D + V_E}\right) \cdot \mathbf{b}^{Debt} \cdot (1 - t_C)$$

Accordingly, the expected return on the assets is given by:

$$\begin{aligned}
(29) \quad E(\tilde{R}_A) &= Rf_E + \frac{Cov(\tilde{R}^A, \tilde{R}_{ME})}{Var(\tilde{R}_{ME})} \Pi_{ME} \\
&= Rf_E + \mathbf{b}_{Assets} \cdot \Pi_{ME} = Rf_D(1 - t_C) + \mathbf{b}_{Assets} \cdot \Pi_{ME}
\end{aligned}$$

where the asset beta, \mathbf{b}_{Assets} , is estimated relative to the *equity* market portfolio. ||

Thus, the risk-free benchmark return to determine expected returns on assets is the same as the equity risk-free benchmark—the after-*corporate*-tax risk-free return on debt securities. The debt and equity betas are both estimated relative to the *equity* market portfolio. Hence, the estimated debt beta is “inflated” by the relative personal-tax inferiority of debt income compared to the beta that would have been estimated against the return of the *debt* market portfolio. Accordingly, when averaged with the estimated equity beta to estimate the asset beta, the inflated estimated beta of the debt is deflated back by $(1 - t_C)$, which in an EME reflects the relative personal-tax disadvantage of debt.

It follows that the intercept of the SML for the firm’s free cash flows is the same as that calculated by Ruback (1986). Proposition 6 generalizes this result for all patterns of asset, debt, and equity cash flows, and it clarifies that this unlevering of the betas is conditional on the existence of an EME.

5. Concluding remarks

There is a disturbing lack of uniformity in the finance profession’s treatment of the effects of capital structure decisions (e.g., the APV and the WACC) and of the effect of the same tax structures on risk-return tradeoffs. For example, a traditional textbook approach first teaches students that the WACC is $r_E * \frac{E}{E+D} + r_D * \frac{D}{E+D} * (1 - t_C)$, where E and D denote the market values of the firm’s equity and debt, r_E and r_D are the expected costs of the firm’s equity and debt, and t_C is the marginal corporate tax rate. Students are then taught the SML, which is then used to calculate the cost of equity,

$r_E = r_f + \mathbf{b}_E * [E(r_M) - r_f]$. There is usually very little discussion of the application of the CAPM to the cost of debt, although consistency requires that a similar risk-return relation should also apply to the expected returns of risky debt. This is indeed implicitly assumed in the many equations to “unlever” the separate beta estimates of equity and debt to an asset beta.

In teaching the SML, little or no attention is paid to the differential taxation of equity and debt income and to the effect that this may have on the risk-free returns and the risk premiums of debt and equity securities. For example, in applying the SML to determine the risk-adjusted return of debt and equity the risk-free rate is often taken to be the same for both security types, which is typically inconsistent with differential taxation of debt and equity income.

Undoubtedly, part of the reason for the lack of an accepted paradigm to treat risk in a world of differential taxation is the perception that in such a world, because of the inherent market segmentation, there is no way to get simple relations between risk and return. In this paper, we overcome this difficulty and provide a framework in which one can analyze risk-return relations even though equity and debt incomes are differentially taxed. In this framework, the debt and equity markets are segmented, which means that the risk-free benchmarks in these markets are generically different. Still, we are able to provide a consistent approach that combines the CAPM and the WACC. We provide necessary and sufficient conditions for the price of risk to be the same in both markets, despite the fact that they are segmented by differential taxation. Under these conditions, we are able to show that the separate debt and equity markets’ SMLs are:

$$E(\tilde{R}_E) = Rf_D * \frac{(1-t_D)}{(1-t_D)} + \mathbf{b}_E * \left[E(\tilde{R}_{ME}) - Rf_D * \frac{(1-t_D)}{(1-t_D)} \right]$$

$$E(\tilde{R}_D) = Rf_D + \mathbf{b}_D * \left[E(\tilde{R}_{ME}) - Rf_D * \frac{(1-t_D)}{(1-t_D)} \right]$$

where t_D and t_E are the marginal tax rates that equate the after-personal-tax returns on debt and equity securities for the marginal investor of each state of nature. We show that the ratio $(1-t_D)/(1-t_E)$ must

be the same in all states for the price of risk to be the same in the segmented equity and debt markets. If this is true, it is also possible to estimate both the debt and equity betas with respect to an equity market portfolio:

$$\mathbf{b}_E = \frac{Cov(\tilde{R}_E, \tilde{R}_{ME})}{Var(\tilde{R}_{ME})} \text{ and } \mathbf{b}_D = \frac{Cov(\tilde{R}_D, \tilde{R}_{ME})}{Var(\tilde{R}_{ME})}$$

Because, in this case, the risk premium that investors attach to both securities types are the same (even though the risk-free rates are different), it is also possible to “unlever” the beta estimates of equity and debt to estimate an asset beta. Interestingly, we also show that the same condition is necessary and sufficient for the standard application of the APV, where the tax effect of leverage on firm value is taken to be some fraction of the value of the firm’s debt.

The necessary and sufficient condition for the price of risk to be the same in the debt and equity markets and for the application of the standard APV approach may seem very restrictive in differentially taxed markets. Yet, extending Miller’s (1977) idea by allowing firms to adjust their menu of debt and equity offering, we find that these conditions are naturally satisfied in the resulting equilibrium. In the extended Miller model, corporations adjust their debt and equity supply until $(1-t_D) = (1-t_E)(1-t_C)$ in each state of the world. In this case, the debt and equity SMLs can be restated in terms of the corporate tax rate:

$$E(\tilde{R}_E) = Rf_D * (1-t_C) + \mathbf{b}_E * [E(\tilde{R}_{ME}) - Rf_D * (1-t_C)]$$

$$E(\tilde{R}_D) = Rf_D + \mathbf{b}_D * [E(\tilde{R}_{ME}) - Rf_D * (1-t_C)]$$

When this is true, a weighted-average asset beta, \mathbf{b}_{assets} , can be used to compute the WACC:

$$WACC = Rf_D * (1-t_C) + \mathbf{b}_{Assets} * [E(\tilde{R}_{ME}) - Rf_D * (1-t_C)]$$

$$\text{where } \mathbf{b}_{Assets} = \mathbf{b}_E \cdot \frac{E}{E+D} + \mathbf{b}_D \cdot (1-t_C) \cdot \frac{E}{E+D}$$

In sum, our analysis provides a setting in which risk-return relations can be analyzed even though debt and equity are differentially taxed. We show what are the necessary and the sufficient conditions for the applicability of the standard tools of corporate finance. We also show how a reasonable equilibrium structure can lead to fulfillment of these conditions. Under these equilibrium conditions we convert the theoretical risk-return relations to practical, estimable relations.

References

- Aivazian, Varouj A. and Jeffrey L. Callen (1987). "Miller's Irrelevance Mechanism: A Note." *Journal of Finance* 42, 169-180.
- Allen, Franklin and Douglas Gale (1991). "Arbitrage, Short Sales, And Financial Innovation," *Econometrica* 59, 1041-1068.
- Allen, Franklin and Douglas Gale (1988). "Optimal Security Design," *Review of Financial Studies* 1, 229-263.
- Auerbach, A. J. and M. King (1983). "Taxation, Portfolio Choice and Debt-Equity Ratios: A General Equilibrium Model." *Quarterly Journal of Economics* 98, 587-609.
- Basak, Suleiman and Michael Gallmeyer (1999). "Capital Market Equilibrium with Differential Taxation." working paper, Wharton School.
- Beja, Avraham (1972). "On Systematic and Unsystematic Components of Financial Risk." *Journal of Finance* 27, 37-45.
- Benninga, Simon and Oded Sarig (1997). *Corporate Finance: A Valuation Approach*, McGraw-Hill, New York, NY.
- Black, Fisher (1974). "International Capital Market Equilibrium with Investment Barriers." *Journal of Financial Economics* 1, 337-352.
- Bossaerts, Peter and Robert M. Dammon (1994). "Tax-Induce Intertemporal Restrictions on Security Returns." *Journal of Finance* 44, 1347-1371.
- Bosshardt, Donald (1984). "Equity Clienteles, Short-Sale Restrictions, and the Miller Equilibrium," *Financial Review* 19-2, 242-250.
- Constantinides, George M. (1983). "Capital Market Equilibrium with Personal Tax." *Econometrica* 51, 611-636.
- Constantinides, George M. (1984). "Optimal Stock Trading with Personal Taxes." *Journal of Financial Economics* 13, 65-89.
- Dammon, Robert M. (1988). "A Security Market and Capital Structure Equilibrium under Uncertainty with Progressive Income Taxes." *Research in Finance* 7, 53-74.
- Dammon, Robert M., Kenneth B. Dunn, and Chester S. Spatt (1989). "A Reexamination of the Value of Tax Options." *Review of Financial Studies* 2, no. 3, 341-372.
- Dammon, Robert M. and Richard C. Green (1987). "Tax Arbitrage and the Existence of Equilibrium Prices for Financial Assets." *Journal of Finance* 42, 1143-1166.
- Dammon, Robert M. and Chester S. Spatt (1996). "The Optimal Trading and Pricing of Asymmetric Capital Gains Taxes and Transaction Costs." *Review of Financial Studies* 9, 921-952.
- DeAngelo, Harry and Ronald W. Masulis (1980). "Optimal Capital Structure under Corporate and Personal Taxation." *Journal of Financial Economics* 8, 3-29.
- Dybvig, Philip and Stephen Ross (1986), "Tax Clienteles and Asset Pricing." *Journal of Finance* 41, 751-771.
- Elton, Edwin J. and Martin J. Gruber (1978). "Taxes and Portfolio Composition." *Journal of Financial Economics* 6, 399-410.
- Gilson, Stuart C., Edith S. Hotchkiss, and Richard S. Ruback (1998). "Valuation of Bankrupt Firms." working paper, Harvard Business School.

- Gordon, Roger H. and Hal R. Varian (1989). "Taxation of Asset Income in the Presence of a World Securities Market." *Journal of International Economics* 26, 205-226.
- Green, Richard C. and Burton Hollifield (1999). "The Personal Tax Advantages of Equity." Working paper, Carnegie Mellon University.
- Hamada, Robert (1969). "Portfolio analysis, Market Equilibrium and Corporation Finance." *Journal of Finance* 24, 13-31.
- Hamada, Robert S. (1972). "The Effect Of The Firm's Capital Structure On The Systematic Risk Of Common Stocks," *Journal of Finance* 27, 435-452.
- Hamada, Robert S. and Myron S. Scholes (1985). "Taxes and Corporate Financial Management," in *Recent Advances in Corporate Finance* (Edward I. Altman and Marti G. Subrahmanyam, editors). Homewood, Illinois: Richard D. Irwin.
- Kaplan, Steven N. and Richard S. Ruback (1995). "The Valuation Of Cash Flow Forecasts: An Empirical Analysis," *Journal of Finance* 50, 1059-1093.
- Litzenberger, Robert H. and Krishna Ramaswamy (1979). "The Effect of Personal Taxes and Dividends on Capital Asset Prices." *Journal of Financial Economics* 7, 163-195.
- Long, John (1977). "Efficient Portfolio Choice with Differential Taxation of Dividends and Capital Gains." *Journal of Financial Economics* 5, 25-53.
- Long, John B. Jr. (1977). "Efficient Portfolio Choice with Differential Taxation of Dividends and Capital Gains." *Journal of Financial Economics* 5, 25-53.
- Miller, Merton H. (1977). "Debt and Taxes." *Journal of Finance* 32, 261-275.
- Modigliani, Franco, and Merton H. Miller (1963). "Corporate Income Taxes and the Cost of Capital: A Correction." *American Economic Review* 53, 433-493.
- Myers, Stewart C. and Richard S. Ruback (1992). "Discounting Rules for Risky Assets." unpublished working paper.
- Ross, Stephen (1987). "Arbitrage and Martingales with Taxation." *Journal of Political Economy* 95, 371-393.
- Ross, Stephen A., Randolph W. Westerfield, and Jeffrey F. Jaffee (1993). *Corporate Finance*. Homewood, IL: Irwin.
- Ruback, Richard S. (1986). "Calculating The Market Value Of Riskless Cash Flows," *Journal of Financial Economics* 15, 323-340.
- Schaefer, Steven (1982). "Taxes and Security Market Equilibrium," in *Financial Economics: Essays in Honor of Paul Cootner*, William Sharpe and Cathryn Cootner (eds.), Prentice Hall, Englewood Cliffs, NJ.
- Sick, Gordon A. (1990). "Tax-Adjusted Discount Rates," *Management Science* 36, 1432-1450.
- Singer, Ronald F. (1979). "Endogenous Marginal Income Tax Rates, Investor Behavior and the Capital Asset Pricing Model," *Journal of Finance* 34, 609-616.
- Taggart, Robert A. (1991). "Consistent Valuation and Cost of Capital Expressions with Corporate and Personal Taxes." *Financial Management* 20 (Autumn), 8-20.
- Talmor, Eli (1989). "Tax Arbitrage Restrictions and Financial Leverage Clienteles." *Journal of Banking*